MCV4U: Calculus & Vectors

Derivatives of General Exponential and Logarithmic Functions

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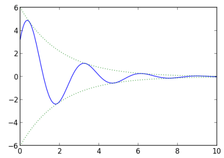

Derivatives Involving $\ln x$

Recap

Determine any value(s) of x where the tangent to $f(x) = 2x \cdot \ln^3 x$ is horizontal.

Use the product and chain rules to find the derivative.

$$\begin{aligned} f'(x) &= 2\ln^3 x + 2x \cdot 3\ln^2 x \cdot \frac{1}{x} \\ &= 2\ln^3 x + 6\ln^2 x \\ &= 2\ln^2 x\,(\ln x + 3) \end{aligned}$$

Derivatives Involving $\ln x$

Since the tangent is horizontal, the slope is zero. Solve each factor.

$$\begin{aligned} 2\ln^2 x &= 0 \\ \ln^2 x &= 0 \\ \ln x &= 0 \\ x &= 1 \end{aligned} \qquad \begin{aligned} \ln x + 3 &= 0 \\ \ln x &= -3 \\ x &= e^{-3} \\ x &= \tfrac{1}{e^3} \end{aligned}$$

Therefore, the tangent is horizontal when $x = 1$ or when $x = \frac{1}{e^3}$.

Derivatives of Exponential Functions

To find the derivative of a general exponential function, $f(x) = b^x$ where $b \neq e$, we can a technique similar to that used to find the derivative of $f(x) = \ln x$.

$$\begin{aligned} y &= b^x \\ \ln y &= \ln b^x \\ &= \ln b \cdot x \\ \tfrac{1}{y} \cdot \tfrac{dy}{dx} &= \ln b \\ \tfrac{dy}{dx} &= y \ln b \\ &= b^x \ln b \end{aligned}$$

Derivative of $y = b^x$

If $f(x) = b^x$, then $f'(x) = b^x \ln b$. If $y = b^x$, $\frac{dy}{dx} = b^x \ln b$.

Derivatives of Exponential Functions

Example

Determine the derivative of $f(x) = 5^x$.

This is pretty straightforward.

$$f'(x) = 5^x \ln 5$$

Example

Determine the derivative of $f(x) = 3x^2 \cdot 2^x$.

Use the product rule.

$$\begin{aligned} f'(x) &= 6x \cdot 2^x + 3x^2 \cdot 2^x \ln 2 \\ &= 3x \cdot 2^x\,(2 + x\ln 2) \end{aligned}$$

Derivatives of Exponential Functions

Example

Determine the slope of the tangent to $y = \dfrac{2^x}{x}$ when $x = 4$.

Use the quotient rule.

$$\begin{aligned} \tfrac{dy}{dx} &= \frac{2^x \ln 2 \cdot x - 2^x}{x^2} \\ &= \frac{2^x\,(x\ln 2 - 1)}{x^2} \end{aligned}$$

Thus, $\left.\frac{dy}{dx}\right|_{x=4} = \dfrac{2^4\,(4\ln 2 - 1)}{4^2} = 4\ln 2 - 1$.

Derivatives of Logarithmic Functions

Using implicit differentiation, we can also determine the derivative of a general logarithmic function.

Recall that if $y = \log_b x$, then $b^y = x$.

$$\begin{aligned} \ln b^y &= \ln x \\ y \ln b &= \ln x \\ \ln b \cdot \tfrac{dy}{dx} &= \tfrac{1}{x} \\ \tfrac{dy}{dx} &= \tfrac{1}{x \ln b} \end{aligned}$$

Derivative of $y = \log_b x$

If $f(x) = \log_b x$, then $f'(x) = \frac{1}{x \ln b}$. If $y = \log_b x$, $\frac{dy}{dx} = \frac{1}{x \ln b}$.

Derivatives of Logarithmic Functions

Example

Determine the derivative of $f(x) = 6 \log_2 x + 5x$.

Use the constant multiple and sum rules for derivatives.

$$f'(x) = \frac{6}{x \ln 2} + 5$$

Example

Determine the slope of the tangent to $y = \log(3x + 2)$ when $x = 1$.

Assuming $\log x$ is the common logarithm (base 10) of x, the derivative is $\frac{dy}{dx} = \frac{3}{(3x+2)\ln 10}$. Note the chain rule here.

Therefore, $\left.\frac{dy}{dx}\right|_{x=1} = \frac{3}{5 \ln 10}$.

Derivatives of Logarithmic Functions

Example

At what point on the function $f(x) = \log_5 x$ is the slope $\frac{1}{25}$?

The derivative is $f'(x) = \frac{1}{x \ln 5}$. Set this equal to $\frac{1}{25}$.

$$\begin{aligned} \tfrac{1}{25} &= \tfrac{1}{x \ln 5} \\ x &= \tfrac{25}{\ln 5} \end{aligned}$$

$f(x) = \log_5\left(\frac{25}{\ln 5}\right) = 2 - \log_5(\ln 5)$.

Therefore, the function has a slope of $\frac{1}{25}$ at the point $\left(\frac{25}{\ln 5}, 2 - \log_5(\ln 5)\right)$, or approximately $(15.5, 1.7)$.

Derivatives of Logarithmic Functions

Example

Determine the equation of the line parallel to $y = (\log_3 x)^2$ when $x = 27$, if it passes through the point $\left(6, \frac{5}{\ln 3}\right)$.

Using the chain rule, $\frac{dy}{dx} = 2 \log_3 x \cdot \frac{1}{x \ln 3}$.

Thus, $\left.\frac{dy}{dx}\right|_{x=27} = 2 \log_3 27 \cdot \frac{1}{27 \ln 3} = \frac{2}{9 \ln 3}$.

$$\begin{aligned} \tfrac{5}{\ln 3} &= \tfrac{2}{9 \ln 3}(6) + b \\ \tfrac{5}{\ln 3} &= \tfrac{4}{3 \ln 3} + b \\ b &= \tfrac{11}{3 \ln 3} \end{aligned}$$

The equation of the line is $y = \frac{2}{9 \ln 3}x + \frac{11}{3 \ln 3}$.

Questions?