MCV4U: Calculus & Vectors

Derivative of the Exponential Function, $y = e^x$

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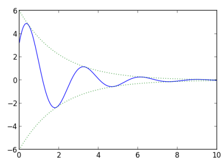


The Number e

Recall that *Euler's number* is an irrational constant, $e \approx 2.71828$.

There are many mathematical definitions of e, including:

e Defined As an Infinite Series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots.$$

When $x = 1$, $e = \sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots.$

A rigorous "proof" of this is not provided, but a table of values for $0 \leq n \leq 10$ should convince us that this is reasonable.

The Number e

n	Approximation of e
0	1.0000000
1	2.0000000
2	2.5000000
3	2.6666666
4	2.7083333
5	2.7166666
6	2.7180555
7	2.7182539
8	2.7182787
9	2.7182815
10	2.7182818

The Number e

e frequently occurs in exponential and logarithmic functions.

Functions involving exponential growth and decay are quite common, and it may be useful to determine the rate of change of such functions.

For instance, a function may model the population of bacteria over time. It may be more important to know how quickly the population is increasing, rather than the population itself at a given time.

Thus, we need the derivative of the exponential function, $y = e^x$.

The Derivative of $y = e^x$

The derivative of $y = e^x$ can be found using first principles.

$$\begin{aligned}\tfrac{dy}{dx} &= \lim_{h\to 0} \frac{e^{x+h} - e^x}{h}\\ &= \lim_{h\to 0} \frac{e^x \cdot e^h - e^x}{h}\\ &= \lim_{h\to 0} \frac{e^x(e^h - 1)}{h}\end{aligned}$$

Since e^x does not depend on h, it can be factored out as a constant.

$$\tfrac{dy}{dx} = e^x \lim_{h\to 0} \frac{e^h - 1}{h}$$

The Derivative of $y = e^x$

Recall that $e^h = \frac{h^0}{0!} + \frac{h^1}{1!} + \frac{h^2}{2!} + \frac{h^3}{3!} + \ldots$, or $e^h = 1 + h + \frac{h^2}{2} + \frac{h^3}{6} + \ldots.$

Substituting this into the derivative,

$$\begin{aligned}\tfrac{dy}{dx} &= e^x \lim_{h\to 0} \frac{[1 + h + \frac{h^2}{2} + \frac{h^3}{6} + \ldots] - 1}{h}\\ &= e^x \lim_{h\to 0} \frac{h + \frac{h^2}{2} + \frac{h^3}{6} + \ldots}{h}\\ &= e^x \lim_{h\to 0} (1 + \tfrac{h}{2} + \tfrac{h^2}{6} + \ldots)\\ &= e^x \cdot \lim_{h\to 0} 1 + \lim_{h\to 0} \tfrac{h}{2} + \lim_{h\to 0} \tfrac{h^2}{6} + \ldots\\ &= e^x\end{aligned}$$

The Derivative of $y = e^x$

Derivative of $y = e^x$

If $y = e^x$, then $\frac{dy}{dx} = e^x$.

This is an interesting result. There are only two functions whose derivatives are the same as the functions.

- $y = ke^x$ where k is a constant, since $\frac{d}{dx}ke^x = k\frac{d}{dx}e^x$
- $y = 0$, since $\frac{d}{dx}0 = 0$

Since the derivative at a point represents the slope of the tangent at that point, the fact that $\frac{d}{dx}ke^x = ke^x$ means that the slope of the tangent to $y = ke^x$ is equals the value of y.

For example, if $x = 0$, $y = e^0 = 1$ and $\left.\frac{dy}{dx}\right|_{x=0} = e^0 = 1$.

The Derivative of $y = e^x$

Example

Determine the derivative of $f(x) = 20e^x$.

Since 20 is a constant multiple, $f'(x) = 20e^x$.

Other derivative rules (addition, product, chain, etc.) can be used to differentiate functions involving e^x.

Note that if a function contains a term involving e^u for some function u, then the chain rule should be used where u is the inner function and e^u the outer function.

The Derivative of $y = e^x$

Example

Determine the derivative of $f(x) = 2x^3 \cdot e^x$.

Use the product rule here.

$$\begin{aligned} f'(x) &= 6x^2 \cdot e^x + 2x^3 \cdot e^x \\ &= 2e^x x^2(3 + x) \end{aligned}$$

Example

Determine the derivative of $y = 7e^{2x}$.

Use the chain rule where $y = 7e^u$ and $u = 2x$.

$$\begin{aligned} \frac{dy}{dx} &= 7e^{2x}(2) \\ &= 14e^{2x} \end{aligned}$$

The Derivative of $y = e^x$

Example

Determine any points where the tangent to $f(x) = x^2 \cdot e^{3x}$ is horizontal.

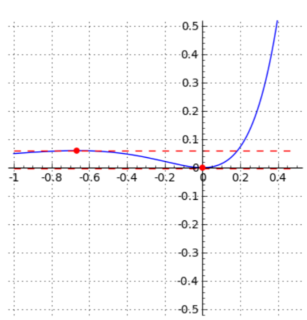
$$\begin{aligned} f'(x) &= 2x \cdot e^{3x} + x^2 \cdot e^{3x} \cdot 3 \\ &= e^{3x}x(3x + 2) \end{aligned}$$

Therefore, the tangent is horizontal when $x = 0$ or $x = -\frac{2}{3}$.

$$f(0) = 0 \qquad f\left(-\tfrac{2}{3}\right) = \left(-\tfrac{2}{3}\right)^2 \cdot e^{3\left(-\frac{2}{3}\right)} = \frac{4}{9e^2}$$

The tangent is horizontal at $(0, 0)$ and $\left(-\frac{2}{3}, \frac{4}{9e^2}\right)$.

The Derivative of $y = e^x$



Questions?