MCV4U: Calculus & Vectors

A Curve Sketching Algorithm

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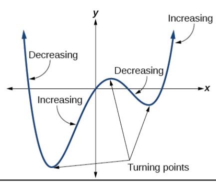


Curve Sketching

Throughout this unit, we have reviewed and introduced many techniques that are useful for making accurate sketches of functions.

While some points (extrema, points of inflection) were not covered in previous courses, we can use calculus to find these points.

A sequence of steps to achieve a result is often referred to as an *algorithm*.

While there is no "one true algorithm" for sketching curves, we can incorporate the following steps to produce graphs accurate enough for our purposes.

Curve Sketching

Algorithm For Curve Sketching

- Use family and end behaviour to determine overall shape (test for symmetry)
- Determine any intercepts for the function (factor)
- Determine any vertical, horizontal, or other asymptotes
- Determine any other discontinuities (holes, jumps, etc.)
- Find critical values, where $f'(x) = 0$ or $f'(x)$ is undefined
- Find concavity using $f''(x)$, or by testing intervals with $f'(x)$
- Determine any local extrema using above two rules
- Sketch the curve

Curve Sketching

Example

Sketch a graph of $f(x) = \frac{4x}{x^2+1}$.

This is a rational function, and is difficult to predict.

There is an x-intercept at $(0, 0)$, which is also the function's y-intercept.

The function is odd, since $f(-x) = -f(x)$.

$$\begin{aligned} f(-x) &= \frac{4(-x)}{(-x)^2+1} \\ &= -\frac{4x}{x^2+1} \\ &= -f(x) \end{aligned}$$

Thus, the function has point symmetry about the origin.

Curve Sketching

There are no vertical asymptotes, since $x^2 + 1 > 0$.

There is a horizontal asymptote at $y = 0$ when $x \to \infty$ and $x \to -\infty$.

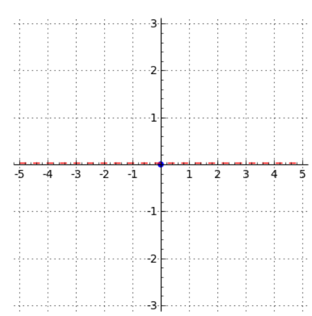
$$\begin{aligned} \lim_{x\to\infty} \frac{4x}{x^2+1} &= \lim_{x\to\infty} \frac{\frac{4x}{x^2}}{\frac{x^2}{x^2}+\frac{1}{x^2}} \\ &= \frac{0}{1+0} \\ &= 0 \end{aligned} \qquad \begin{aligned} \lim_{x\to-\infty} \frac{4x}{x^2+1} &= \lim_{x\to-\infty} \frac{\frac{4x}{x^2}}{\frac{x^2}{x^2}+\frac{1}{x^2}} \\ &= \frac{0}{1+0} \\ &= 0 \end{aligned}$$

There are no oblique asymptotes, since the degree of the numerator is smaller than that of the denominator.

There are no other discontinuities.

Curve Sketching

At this stage, we know very little about the graph.



Curve Sketching

Take the first derivative to find critical points.

$$f'(x) = \frac{4(x^2+1) - 4x(2x)}{(x^2+1)^2} = \frac{4-4x^2}{(x^2+1)^2} = \frac{4(1-x^2)}{(x^2+1)^2}$$

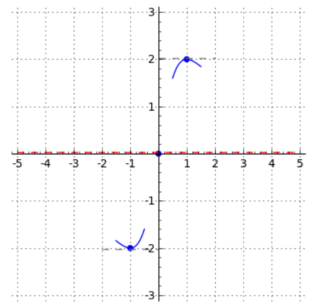
Thus, there are critical values when $1 - x^2 = 0$, or $x = \pm 1$.

Since $f(1) = 2$ and $f(-1) = -2$, add $(1, 2)$ and $(-1, -2)$ on the graph.

Based on the end behaviour of the function, the most likely scenario is that these two points are local extrema.

Curve Sketching

We now know a little bit more, but should check for points of inflection.



Curve Sketching

Take the second derivative to find potential points of inflection and check concavity.

$$f''(x) = \frac{-8x(x^2+1)^2 - 4(1-x^2)(2)(x^2+1)(2x)}{(x^2+1)^4} = \frac{-8x(x^2+1) - 16x(1-x^2)}{(x^2+1)^3} = \frac{-8x^3 - 8x - 16x + 16x^3}{(x^2+1)^3} = \frac{8x^3 - 24x}{(x^2+1)^3} = \frac{8x(x^2-3)}{(x^2+1)^3}$$

Curve Sketching

Thus, there are critical values when $x = 0$ or when $x^2 - 3 = 0$, or $x = \pm\sqrt{3}$.

Since $f(\sqrt{3}) = \sqrt{3}$ and $f(-\sqrt{3}) = -\sqrt{3}$, add $(\sqrt{3}, \sqrt{3})$ and $(-\sqrt{3}, -\sqrt{3})$ to the graph.

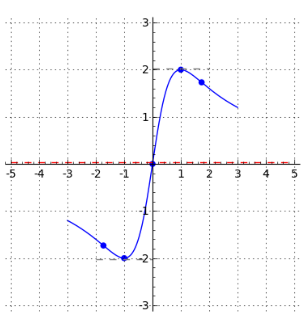
Test intervals for concavity.

Interval	$(-\infty, -\sqrt{3})$	$(-\sqrt{3}, 0)$	$(0, \sqrt{3})$	$(\sqrt{3}, \infty)$
x	-2	-1	1	2
$f''(x)$	$-\frac{16}{125}$	2	-2	$\frac{16}{125}$
sign	$-$	$+$	$-$	$+$

Therefore, $(0, 0)$, $(\sqrt{3}, \sqrt{3})$ and $(-\sqrt{3}, \sqrt{3})$ are all points of inflection.

Curve Sketching

Our graph is starting to take shape now.



Curve Sketching

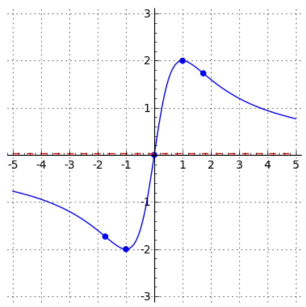
Although it is fairly easy to see that $(1, 2)$ and $(-1, -2)$ must be local extrema, we can verify this using the second derivative (or the first).

In the previous table, we found that $f''(1) = -2 < 0$, so $(1, 2)$ is a local maximum.

Similarly, since $f''(-1) = 2 > 0$, $(-1, -2)$ is a local minimum.

Curve Sketching

Putting everything together, we obtain the following graph of $f(x)$.



Curve Sketching

Example

Sketch a graph of $y = \dfrac{-x^4 + 3x^3 + 3x^2 - 7x - 6}{x - 3}$.

Since $-(3)^4 + 3(3)^3 + 3(3)^2 - 7(3) - 6 = 0$, $x - 3$ is a factor of the numerator.

	-1	3	3	-7	-6
3		-3	0	9	6
	-1	0	3	2	0

Thus, $y = -x^3 + 3x + 2$, $x \neq 3$.

Therefore, there is a hole in the graph at $(3, -16)$, since $y|_{x=3} = -3^3 + 3(3) + 2 = -16$.

Curve Sketching

Since polynomial functions do not have any asymptotes, the only discontinuity occurs at $(3, -16)$.

Other key points are the y-intercept, at $(0, 2)$, and the x-intercepts, which can be found by factoring.

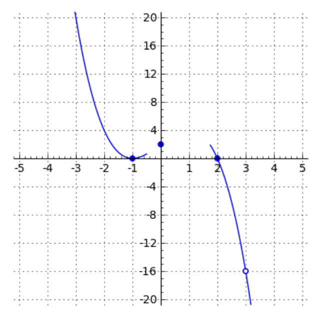
	-1	0	3	2
2		-2	-4	-2
	-1	-2	-1	0

$$\begin{aligned} y &= (x-2)(-x^2 - 2x - 1) \\ &= -(x-2)(x^2 + 2x + 1) \\ &= -(x-2)(x+1)^2 \end{aligned}$$

The x-intercepts are at $(2, 0)$ and $(-1, 0)$.

Curve Sketching

Combining these points with the end behaviour (Q2→Q4) gives us the following graph.



Curve Sketching

Since the x-intercept at $(-1, 0)$ has order 2, it will be a local extremum (a minimum, given the function's end behaviour).

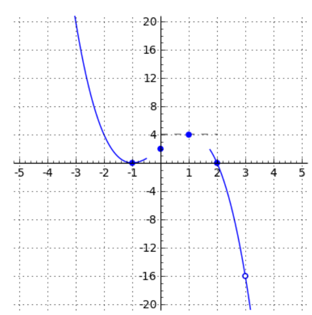
The other extremum (a maximum) will occur somewhere on the interval $(-1, 2)$. Find the derivative to identify any critical points.

$$\begin{aligned} \tfrac{dy}{dx} &= -3x^2 + 3 \\ &= -3(x^2 - 1) \\ &= -3(x+1)(x-1) \end{aligned}$$

Critical points are $x = \pm 1$. Note that one critical point, $x = -1$, was already identified as a local minimum. Thus, the local maximum should occur when $x = 1$, at $y|_{x=1} = -1^3 + 3(1) + 2 = 4$.

Curve Sketching

Add $(1, 4)$ to the graph.



Curve Sketching

Find the second derivative to identify possible points of inflection. There should be one on the interval $(-1, 1)$, since the function moves from a local minimum to a local maximum on that interval.

$$\frac{d^2y}{dx^2} = -6x$$

Since $\frac{d^2y}{dx^2} = 0$ when $x = 0$, there is a point of inflection at the y-intercept, $(0, 2)$.

This can be verified by testing values on either side.

Interval	$(-\infty, 0)$	$(0, \infty)$
x	-1	1
$f''(x)$	6	-6
sign	$+$	$-$

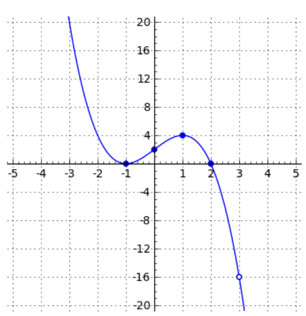
Curve Sketching

Since the function changes from concave up to concave down at $x = 0$, it is a point of inflection.

Note that it also confirms that $(-1, 0)$ is a local minimum, and $(1, 4)$ is a local maximum, since $x = \pm 1$ were critical points.

Curve Sketching

Putting everything together gives us the following graph.



Questions?