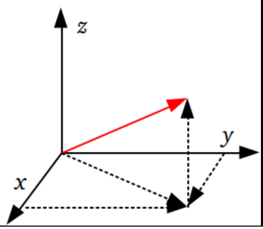
MCV4U: Calculus & Vectors

The Cross Product of Two Vectors

J. Garvin



Cross Product of Geometric Vectors

In addition to the basic operations with vectors (addition, subtraction, scalar multiplication), we can define additional operations, such as the dot product.

Unlike the dot product, the *cross product* is an operation produces a vector rather than a scalar.

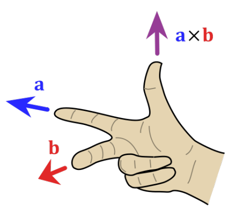
This vector is *orthogonal* (perpendicular) to the two vectors used in the calculation.

Cross Product of Two Geometric Vectors

The cross product of two vectors, $\vec{u}$ and $\vec{v}$, is $\vec{u} \times \vec{v} = (|\vec{u}||\vec{v}| \sin\theta)\hat{n}$, where $\hat{n}$ is a unit vector orthogonal to both $\vec{u}$ and $\vec{v}$.

Cross Product

The direction of $\vec{a} \times \vec{b}$ is determined using the *right-hand-rule*: curl the fingers of your right hand from $\vec{a}$ to $\vec{b}$, and $\vec{a} \times \vec{b}$ will point in the direction of your thumb.



Cross Product of Geometric Vectors

Example

Calculate $\vec{a} \times \vec{b}$ if $|\vec{a}| = 20$, $|\vec{b}| = 35$ and the angle between $\vec{a}$ and $\vec{b}$ is $70°$.

$$\begin{aligned} \vec{a} \times \vec{b} &= (20 \cdot 35 \cdot \sin 70°)\hat{n} \\ &\approx 657.8\hat{n} \end{aligned}$$

Cross Product of Geometric Vectors

Example

Calculate $\vec{a} \times \vec{b}$ if $\vec{a} = (3, 0, 5)$ and $\vec{b} = (-1, 2, -3)$.

Begin by finding the magnitudes of $\vec{a}$ and $\vec{b}$ as before.

$$\begin{aligned} |\vec{a}| &= \sqrt{3^2 + 0^2 + 5^2} \\ &= \sqrt{34} \\ |\vec{b}| &= \sqrt{(-1)^2 + 2^2 + (-3)^2} \\ &= \sqrt{14} \end{aligned}$$

Cross Product of Geometric Vectors

Nex, find the angle between the vectors.

$$\begin{aligned} \vec{a} \cdot \vec{b} &= (3)(-1) + (0)(2) + (5)(-3) \\ &= -18 \\ \theta &= \cos^{-1}\left(\frac{-18}{\sqrt{34}\sqrt{14}}\right) \\ &\approx 145.6° \end{aligned}$$

Finally, calculate the cross product.

$$\begin{aligned} \vec{a} \times \vec{b} &\approx (\sqrt{34} \cdot \sqrt{14} \cdot \sin 145.6°)\hat{n} \\ &\approx 12.3\hat{n} \end{aligned}$$

Cross Product of Algebraic Vectors

The previous examples are not as useful as they could be, as they express $\vec{a} \times \vec{b}$ as scalar multiples of $\hat{n}$.

It would be more useful for us to express $\vec{a} \times \vec{b}$ as a position vector, so that we can work with it later.

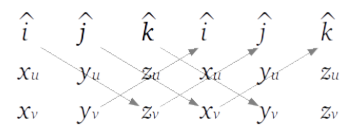
Fortunately, there is a formula for doing this.

Cross Product of Algebraic Vectors

Cross Product of Two Algebraic Vectors

Let $\vec{u} = (x_u, y_u, z_u)$ and $\vec{v} = (x_v, y_v, z_v)$. Then $\vec{u} \times \vec{v} = (y_u z_v - z_u y_v, z_u x_v - x_u z_v, x_u y_v - y_u x_v)$.

One way to remember this is to use the following diagram.

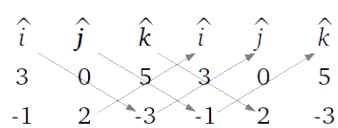


Note that for each entry, $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$, the diagram makes a "cross" containing the four entries that make up that component of the final vector $\vec{u} \times \vec{v}$.

Cross Product of Algebraic Vectors

Example

Calculate $\vec{a} \times \vec{b}$ if $\vec{a} = (3, 0, 5)$ and $\vec{b} = (-1, 2, -3)$.

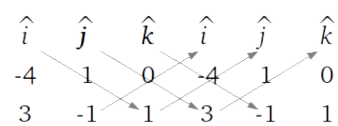


$$\begin{aligned} \vec{a} \times \vec{b} &= (0(-3) - 5(2), 5(-1) - 3(-3), 3(2) - 0(-1)) \\ &= (-10, 4, 6) \end{aligned}$$

Cross Product of Algebraic Vectors

Example

If $\vec{p} = (-4, 1, 0)$ and $\vec{q} = (3, -1, 1)$, calculate $\vec{p} \times \vec{q}$.

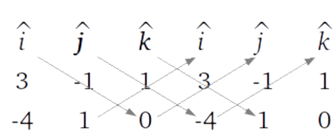


$$\begin{aligned} \vec{p} \times \vec{q} &= (1(1) - 0(-1), 0(3) - (-4)(1), -4(-1) - 1(3)) \\ &= (1, 4, 1) \end{aligned}$$

Cross Product of Algebraic Vectors

Example

If $\vec{p} = (-4, 1, 0)$ and $\vec{q} = (3, -1, 1)$, calculate $\vec{q} \times \vec{p}$.



$$\begin{aligned} \vec{q} \times \vec{p} &= (-1(0) - 1(1), 1(-4) - 3(0), 3(1) - (-1)(-4)) \\ &= (-1, -4, -1) \end{aligned}$$

Properties of the Cross Product

Notice that $\vec{q} \times \vec{p}$ is the opposite vector of $\vec{p} \times \vec{q}$.

This means that $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$.

Unlike the dot product, the cross product is not commutative.

The cross product is not associative either, and $\vec{u} \times (\vec{v} \times \vec{w}) \neq (\vec{u} \times \vec{v}) \times \vec{w}$.

The cross product *is* distributive over addition, so $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$.

It is also distributive with respect to scalar multiplication, so $k(\vec{u} \times \vec{v}) = k\vec{u} \times \vec{v} = \vec{u} \times k\vec{v}$.

Questions?