MCR3U: Functions

Proving Trigonometric Identities

Part 2: Useful Techniques

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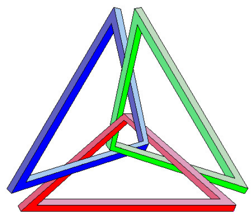

Techniques For Proving Trigonometric Identities

Proving trigonometric identities can be challenging, since there is no fixed set of rules for doing so.

In most cases, you will develop a feel for proving identities by relating them to previously worked examples.

Beyond the basic techniques (writing things in terms of $\sin\theta$ and $\cos\theta$ and simplifying), it may be necessary to use other techniques borrowed from other areas.

Using the Conjugate

Example

Expand and simplify $\dfrac{a}{b+c}\cdot\dfrac{b-c}{b-c}$.

In the example, we say that $b-c$ is the *conjugate* of $b+c$.

What happens when we multiply?

$$\begin{aligned}\frac{a}{b+c}\cdot\frac{b-c}{b-c} &= \frac{a(b-c)}{(b+c)(b-c)}\\ &= \frac{a(b-c)}{b^2-c^2}\end{aligned}$$

Notice that multiplying the binomial in the denominator with its conjugate produced a difference of squares.

Using the Conjugate

Since $1-\sin^2\theta$ and $1-\cos^2\theta$ are differences of squares, we can use the conjugate in conjunction with the Pythagorean Identity to simplify trigonometric expressions.

For example, $1-\sin^2\theta$ can be replaced with $\cos^2\theta$, and $1-\cos^2\theta$ with $\sin^2\theta$.

Often, this is a good approach when dealing with rational expressions that contain different ratios in the numerator and the denominator.

Using the Conjugate

Example

Prove that $\dfrac{\cos\theta}{1+\sin\theta}=\dfrac{1-\sin\theta}{\cos\theta}$.

$$\begin{aligned}\text{LHS} &= \frac{\cos\theta}{1+\sin\theta}\\ &= \frac{\cos\theta}{1+\sin\theta}\cdot\frac{1-\sin\theta}{1-\sin\theta}\\ &= \frac{(\cos\theta)(1-\sin\theta)}{1-\sin^2\theta}\\ &= \frac{(\cos\theta)(1-\sin\theta)}{\cos^2\theta}\\ &= \frac{1-\sin\theta}{\cos\theta}\\ &= \text{RHS}\end{aligned}$$

Factoring Trigonometric Expressions

Another useful technique is to factor trigonometric identities, similar to how other binomials and trinomials are factored.

If a rational expression contains the same factor in both its numerator and its denominator, then that factor can be cancelled out, leaving an equivalent expression.

All of the standard factoring techniques (common, simple, complex, perfect squares, differences of squares) may apply, and more than one method of factoring may be required.

Factoring Trigonometric Expressions

Example

Prove that $\dfrac{\cos^3\theta + \cos^2\theta}{1-\sin^2\theta} = \cos\theta + 1$.

$$\begin{aligned}\text{LHS} &= \frac{\cos^3\theta + \cos^2\theta}{1-\sin^2\theta}\\ &= \frac{(\cos^2\theta)(\cos\theta+1)}{1-\sin^2\theta}\\ &= \frac{(\cos^2\theta)(\cos\theta+1)}{\cos^2\theta}\\ &= \cos\theta + 1\\ &= \text{RHS}\end{aligned}$$

Factoring Trigonometric Expressions

Example

Prove that $\dfrac{\sin^2\theta - \cos^2\theta}{\cos\theta - \sin\theta} = -\sin\theta - \cos\theta$.

$$\begin{aligned}\text{LHS} &= \frac{\sin^2\theta - \cos^2\theta}{\cos\theta - \sin\theta}\\ &= \frac{(\sin\theta+\cos\theta)(\sin\theta-\cos\theta)}{\cos\theta-\sin\theta}\\ &= -\frac{(\sin\theta+\cos\theta)(\cos\theta-\sin\theta)}{\cos\theta-\sin\theta}\\ &= -(\sin\theta+\cos\theta)\\ &= -\sin\theta-\cos\theta\\ &= \text{RHS}\end{aligned}$$

Disproving Trigonometric Identities

Disproving a trigonometric identity can be done in two ways:

- Determine a counter-example that shows that the equality does not hold (requires choosing an appropriate value of θ.
- Determine an equivalent expression for the LHS that is impossible to be equal to the RHS (requires identities).

Each method has its pros and cons, but both are valid.

Disproving Trigonometric Identities

Example

Show that $\sin^2\theta - \cos^2\theta \neq 1 + 2\cos^2\theta$.

Use a value of $\theta = 0°$ to evaluate the LHS and RHS.

$$\begin{aligned}\text{LHS} &= \sin^2 0° - \cos^2 0°\\ &= 0 - 1\\ &= -1\\ \text{RHS} &= 1 + 2\cos^2 0°\\ &= 1 + 2(1)^2\\ &= 3\end{aligned}$$

Since LHS $\neq$ RHS, $\sin^2\theta - \cos^2\theta \neq 1 + 2\cos^2\theta$.

Disproving Trigonometric Identities

Example

Show that $\sin\theta \cdot \cot\theta \neq -\cos\theta$.

Simplify the LHS.

$$\begin{aligned}\text{LHS} &= \sin\theta \cdot \cot\theta\\ &= \sin\theta \cdot \frac{\cos\theta}{\sin\theta}\\ &= \cos\theta\end{aligned}$$

Since $\cos\theta \neq -\cos\theta$ (except for *some* values), $\sin\theta \cdot \cot\theta \neq -\cos\theta$.

Questions?