MCR3U: Functions

Proving Trigonometric Identities

Part 1: Proofs Using Basic Identities

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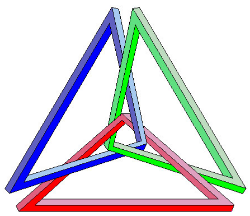


Reciprocal Identities

Recall that the three *primary* trigonometric ratios are $\sin\theta$, $\cos\theta$ and $\tan\theta$.

The three *secondary* ratios are $\csc\theta$, $\sec\theta$ and $\cot\theta$.

Since the secondary ratios are reciprocals of the primary ratios, they are often called the *reciprocal* ratios.

This relationship gives us three *trigonometric identities*.

Reciprocal Identities

For any value of θ, the reciprocal identities are $\csc\theta = \frac{1}{\sin\theta}$, $\sec\theta = \frac{1}{\cos\theta}$ and $\cot\theta = \frac{1}{\tan\theta}$.

These identities can be used to prove that one trigonometric expression is equivalent to another.

Proofs Using Reciprocal Identities

Example

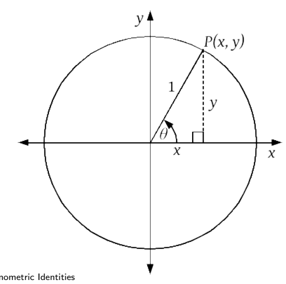
Prove that $\sin\theta \cdot \csc\theta = 1$.

$$\begin{aligned} \text{LHS} &= \sin\theta \cdot \csc\theta \\ &= \sin\theta \cdot \frac{1}{\sin\theta} \\ &= \frac{\sin\theta}{\sin\theta} \\ &= 1 \\ &= \text{RHS} \end{aligned}$$

Pythagorean Identity

Imagine point $P(x, y)$ somewhere on the circumference of a unit circle (radius 1) centred at the origin.

It is possible to construct a right triangle by drawing a vertical line from point P to the x-axis, as shown.

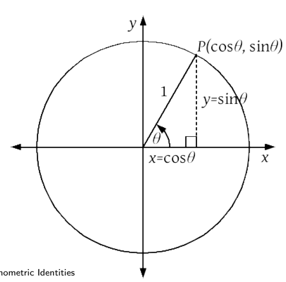


Pythagorean Identity

Since $\sin\theta = \frac{y}{1}$, then $y = \sin\theta$.

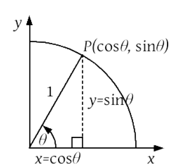
Similarly, $\cos\theta = \frac{x}{1}$, so $x = \cos\theta$.

Therefore, the coordinates of any point P on a unit circle, rotated θ degrees from standard position, are $(\cos\theta, \sin\theta)$.



Pythagorean Identity

Let P be any point on a unit circle.



By the Pythagorean Theorem, $x^2 + y^2 = 1$.

Since $x = \cos\theta$ and $y = \sin\theta$, $x^2 + y^2 = \cos^2\theta + \sin^2\theta = 1$.

Pythagorean Identity

For any value of θ, $\sin^2\theta + \cos^2\theta = 1$.

Proofs Using the Pythagorean Identity

Example

Prove that $\sin^2\theta + 4\cos^2\theta = -3\sin^2\theta + 4$.

$$\begin{aligned} \text{LHS} &= \sin^2\theta + 4\cos^2\theta \\ &= \sin^2\theta + 4(1-\sin^2\theta) \\ &= \sin^2\theta + 4 - 4\sin^2\theta \\ &= -3\sin^2\theta + 4 \\ &= \text{RHS} \end{aligned}$$

Proofs Using the Pythagorean Identity

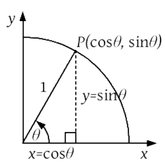
Example

Prove that $\dfrac{1+\sin^2\theta-\cos^2\theta}{2\sin\theta} = \sin\theta$.

$$\begin{aligned} \text{LHS} &= \frac{1+\sin^2\theta-\cos^2\theta}{2\sin\theta} \\ &= \frac{\sin^2\theta+\sin^2\theta}{2\sin\theta} \\ &= \frac{2\sin^2\theta}{2\sin\theta} \\ &= \sin\theta \\ &= \text{RHS} \end{aligned}$$

Tangent and Cotangent Identities

Let P be any point on a unit circle.



Since $\tan\theta = \frac{y}{x}$, $\tan\theta = \frac{\sin\theta}{\cos\theta}$.

Since $\cot\theta = \frac{1}{\tan\theta}$, $\cot\theta = \frac{\cos\theta}{\sin\theta}$.

Tangent and Cotangent Identities

For any value of θ, $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ and $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$.

Proofs Using Tangent and Cotangent Identities

Example

Prove that $\sin\theta \cdot \cot\theta = \cos\theta$.

$$\begin{aligned} \text{LHS} &= \sin\theta \cdot \cot\theta \\ &= \sin\theta \cdot \frac{\cos\theta}{\sin\theta} \\ &= \cos\theta \\ &= \text{RHS} \end{aligned}$$

General Rules For Proving Identities

While there is no specific procedure for proving a trigonometric identity, the following general rules may help.

- Try to simplify, rather than expand, when possible.
- Replace all instances of $\tan\theta$, $\cot\theta$, $\sec\theta$ and $\csc\theta$ with $\sin\theta$ and $\cos\theta$.
- DO NOT "move" terms or factors across an = sign, since this presupposes that the identity is true.
- If necessary, work on both sides of an identity simultaneously and meet somewhere in the middle.

Proofs Using the Basic Identities

Example

Prove that $\cos\theta \cdot \cot\theta + \csc\theta = \dfrac{\cos^2\theta+1}{\sin\theta}$.

$$\begin{aligned} \text{LHS} &= \cos\theta \cdot \cot\theta + \csc\theta \\ &= \cos\theta \cdot \frac{\cos\theta}{\sin\theta} + \frac{1}{\sin\theta} \\ &= \frac{\cos^2\theta}{\sin\theta} + \frac{1}{\sin\theta} \\ &= \frac{\cos^2\theta+1}{\sin\theta} \\ &= \text{RHS} \end{aligned}$$

Proofs Using the Basic Identities

Example

Prove that $\sec^2\theta - 1 = \tan^2\theta$.

$$\begin{aligned} \text{LHS} &= \sec^2\theta - 1 \\ &= \frac{1}{\cos^2\theta} - \frac{\cos^2\theta}{\cos^2\theta} \\ &= \frac{1-\cos^2\theta}{\cos^2\theta} \\ &= \frac{\sin^2\theta}{\cos^2\theta} \\ &= \tan^2\theta \\ &= \text{RHS} \end{aligned}$$

Questions?