MCR3U: Functions

Solving Quadratic Equations

Part 1: Factoring and Quadratic Formula

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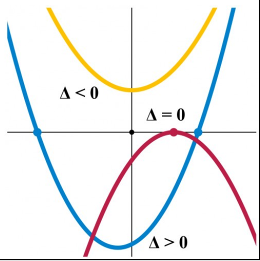

Solving Quadratic Equations

Recall that to *solve* a quadratic equation means to find all values of the independent variable to satisfy a given equation.

For example, the quadratic equation $x^2 + 5x + 6 = 0$ has two solutions, $x = -2$ or $x = -3$.

There are several techniques that we can use to solve quadratic equations.

Solving by Factoring

Often, the easiest method of solving a quadratic equation is to factor it.

We have reviewed several methods of factoring for:

- simple trinomials (inspection)
- complex trinomials (decomposition)
- perfect squares ($b = 2\sqrt{a}\sqrt{c}$)
- differences of squares (inspection)
- grouping

Solving by Factoring

Example

Solve $10x^2 + 13x - 3 = 0$.

$$\begin{aligned} 10x^2 + 13x - 3 &= 0 \\ 10x^2 + 15x - 2x - 3 &= 0 \\ 5x(2x+3) - 1(2x+3) &= 0 \\ (5x-1)(2x+3) &= 0 \\ x &= \frac{1}{5} \text{ or } -\frac{3}{2} \end{aligned}$$

Solving by Factoring

Your Turn

Solve $45x^3 - 5x = 0$.

$$\begin{aligned} 45x^3 - 5x &= 0 \\ 5x(9x^2 - 1) &= 0 \\ 5x(3x-1)(3x+1) &= 0 \\ x &= 0, \frac{1}{3} \text{ or } -\frac{1}{3} \end{aligned}$$

Solving by Completing the Square

Sometimes it is not easy to factor a quadratic equation.

The equation $x^2 + 4x + 2 = 0$ has solutions $-2 + \sqrt{2}$ and $-2 - \sqrt{2}$. These are not "obvious" in any way.

Another technique that can be used to solve quadratic equations is Completing the Square.

While we typically associate this technique with converting a quadratic from standard to vertex form, this form is easy to isolate the independent variable and, thus, solve for any intercepts.

Solving by Completing the Square

Example

Determine the x-intercepts of $f(x) = 2x^2 + 12x + 13$.

$$\begin{aligned} f(x) &= 2x^2 + 12x + 13 \\ &= 2(x^2 + 6x) + 13 \\ &= 2(x^2 + 6x + 9 - 9) + 13 \\ &= 2(x+3)^2 - 5 \end{aligned}$$

Solving by Completing the Square

Now that the function is in vertex form, isolate x.

$$\begin{aligned} 0 &= 2(x+3)^2 - 5 \\ 5 &= 2(x+3)^2 \\ \frac{5}{2} &= (x+3)^2 \\ \pm\sqrt{\frac{5}{2}} &= x + 3 \\ -3 \pm \sqrt{\frac{5}{2}} &= x \end{aligned}$$

The zeroes of $f(x)$ are $-3 + \sqrt{\frac{5}{2}}$ and $-3 - \sqrt{\frac{5}{2}}$.

Solving by Completing the Square

Completing the square can be a tedious process, prone to errors.

For this reason, we develop an explicit formula that can determine the zeroes of a quadratic function in standard form, $f(x) = ax^2 + bx + c$.

The formula itself comes directly from completing the square.

Solving Using the Quadratic Formula

The Quadratic Formula

The zeroes of a quadratic function, $f(x) = ax^2 + bx + c$, are given by $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

$$\begin{aligned} ax^2 + bx + c &= 0 \\ a\left(x^2 + \frac{b}{a}x\right) &= -c \\ a\left(x^2 + \frac{b}{a} + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}\right) &= -c \\ a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} &= -c \end{aligned}$$

Solving Using the Quadratic Formula

$$\begin{aligned} a\left(x + \frac{b}{2a}\right)^2 &= -c + \frac{b^2}{4a} \\ \left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} \\ x + \frac{b}{2a} &= \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \\ x &= -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \\ x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \end{aligned}$$

Solving Using the Quadratic Formula

Example

Determine the zeroes of $g(x) = 2x^2 + 8x + 3$.

$$\begin{aligned} x &= \tfrac{-8 \pm \sqrt{8^2 - 4(2)(3)}}{2(2)} \\ &= \tfrac{-8 \pm \sqrt{64 - 24}}{4} \\ &= \tfrac{-8 \pm \sqrt{40}}{4} \\ &= \tfrac{-8 \pm 2\sqrt{10}}{4} \\ &= \tfrac{-4 \pm \sqrt{10}}{2} \end{aligned}$$

The zeroes of $g(x)$ are $x = \frac{-4 + \sqrt{10}}{2}$ and $x = \frac{-4 - \sqrt{10}}{2}$.

Solving Using the Quadratic Formula

Your Turn

Determine the zeroes of $v(t) = -9t^2 + 6t + 1$.

$$\begin{aligned} t &= \frac{-6 \pm \sqrt{6^2 - 4(-9)(1)}}{2(-9)} \\ &= \frac{-6 \pm \sqrt{36+36}}{-18} \\ &= \frac{-6 \pm \sqrt{72}}{-18} \\ &= \frac{-6 \pm 6\sqrt{2}}{-18} \\ &= \frac{1 \pm \sqrt{2}}{3} \end{aligned}$$

The zeroes of $v(t)$ are $t = \frac{1+\sqrt{2}}{3}$ and $t = \frac{1-\sqrt{2}}{3}$.

Solving Using the Quadratic Formula

Example

Determine the zeroes of $f(x) = 2x^2 - 12x + 23$.

$$\begin{aligned} x &= \frac{12 \pm \sqrt{(-12)^2 - 4(2)(23)}}{2(2)} \\ &= \frac{12 \pm \sqrt{144 - 184}}{4} \\ &= \frac{12 \pm \sqrt{-40}}{4} \end{aligned}$$

Since the square root of a negative number is not defined in the real number system, $f(x)$ has no zeroes.

The Number/Nature of the Zeroes

We do not need to use the entire quadratic formula to determine this. The section inside of the square root is sufficient.

The *discriminant* of the quadratic formula is $b^2 - 4ac$, and can be used to identify the number, and nature, of the zeroes of a quadratic function.

Using the Discriminant

Given a quadratic function in standard form, $f(x) = ax^2 + bx + c$, then the function has:

- 2 distinct real roots when $b^2 - 4ac > 0$
- 1 repeated real root when $b^2 - 4ac = 0$
- 2 distinct complex roots when $b^2 - 4ac < 0$

The Number/Nature of the Zeroes

In the first case, the "plus or minus" in the quadratic equation ensures that there are two distinct answers.

In the second case, adding or subtracting the square root of zero results in the same answer.

In the last case, there are no real roots because of the negative inside of the square root.

When we are only interested in how many real roots there are, or whether they are real or complex, we can use the discriminant instead of the quadratic formula.

The Number/Nature of the Zeroes

Example

State the number, and nature, of the zeroes of $g(x) = -5x^2 + 2x + 8$.

The discriminant is $2^2 - 4(-5)(8) = 164$, so $g(x)$ has two distinct real roots.

Example

State the number, and nature, of the zeroes of $h(x) = 4x^2 - 12x + 9$.

The discriminant is $(-12)^2 - 4(4)(9) = 0$, so $h(x)$ has one repeated real root.

Note that $h(x) = (2x - 3)^2$, with a vertex at $\left(\frac{3}{2}, 0\right)$. The vertex is the parabola's only x-intercept.

Questions?