MCR3U: Functions

Rational Exponents

J. Garvin

Exponent Laws

Recap

Complete the table below, and comment on any patterns.

2^3	8
2^2	4
2^1	2
2^0	1
2^{-1}	$\frac{1}{2}$
2^{-2}	$\frac{1}{4}$

As we move down the table, each previous value is divided by two (or multiplied by one half).

Exponent Laws

From the table, it is easy to see why $2^0 = 1$, since dividing the previous value by two yields $\frac{2^1}{2} = \frac{2}{2} = 1$.

It is also evident that $2^{-1} = \frac{1}{2}$, since $\frac{2^0}{2} = \frac{1}{2}$.

What would happen if the exponent was a rational number, such as $2^{\frac{1}{2}}$?

Based on the table, $2^{\frac{1}{2}}$ should be some value between 1 and 2, since $0 < \frac{1}{2} < 1$.

Is this value 1.5 (the midpoint of 1 and 2), or some other value instead?

Exponent Laws For Rational Exponents

To get an idea of how rational exponents work, consider the following example:

$$\begin{aligned} 2^{\frac{1}{2}} \cdot 2^{\frac{1}{2}} &= \left(2^{\frac{1}{2}}\right)^2 \\ &= 2^1 \\ &= 2 \end{aligned}$$

Thus, $2^{\frac{1}{2}}$ squared is 2. Is there another number with this property?

$$\begin{aligned} \sqrt{2} \cdot \sqrt{2} &= \left(\sqrt{2}\right)^2 \\ &= 2 \end{aligned}$$

Therefore, $2^{\frac{1}{2}} = \sqrt{2}$.

Exponent Laws For Rational Exponents

We can generalize this for any base x, such that $x \geq 0$ (if x is negative, then $\sqrt{x}$ is undefined).

$$\begin{aligned} x^{\frac{1}{2}} \cdot x^{\frac{1}{2}} &= \left(x^{\frac{1}{2}}\right)^2 \\ &= x^1 \\ &= x \end{aligned}$$

$$\begin{aligned} \sqrt{x} \cdot \sqrt{x} &= \left(\sqrt{x}\right)^2 \\ &= x \end{aligned}$$

Therefore, $x^{\frac{1}{2}} = \sqrt{x}$, $x \geq 0$.

Exponent Laws For Rational Exponents

Further generalizing, let the exponent be $\frac{1}{n}$. Then,

$$\begin{aligned} \underbrace{x^{\frac{1}{n}} \cdot x^{\frac{1}{n}} \cdot \ldots \cdot x^{\frac{1}{n}}}_{n \text{ times}} &= \left(x^{\frac{1}{n}}\right)^n \\ &= x^1 \\ &= x \end{aligned}$$

$$\begin{aligned} \underbrace{\sqrt[n]{x} \cdot \sqrt[n]{x} \cdot \ldots \cdot \sqrt[n]{x}}_{n \text{ times}} &= \left(\sqrt[n]{x}\right)^n \\ &= x \end{aligned}$$

Exponent Laws For Rational Exponents

This gives us the first exponent law for rational exponents.

Exponent Law When Numerator = 1

For any value x, $x^{\frac{1}{n}} = \sqrt[n]{x}$, provided that $x \geq 0$ if n is even.

While it is not possible to take the square root (or fourth root, sixth root, etc.) of a negative value, it is possible using odd-valued roots.

For instance, $\sqrt{-8}$ is not a real number, but $\sqrt[3]{-8} = -2$, since $(-2)^3 = -8$.

Exponent Laws For Rational Exponents

Example

Write $3^{\frac{1}{2}}$ in radical form.

$3^{\frac{1}{2}} = \sqrt{3}$.

Example

Write $\left(\frac{7}{4}\right)^{\frac{1}{2}}$ in radical form.

$\left(\frac{7}{4}\right)^{\frac{1}{2}} = \frac{\sqrt{7}}{\sqrt{4}} = \frac{\sqrt{7}}{2}$.

Exponent Laws For Rational Exponents

Example

Write $\sqrt[6]{x}$ in exponential form.

$\sqrt[6]{x} = x^{\frac{1}{6}}$.

Example

Write $\sqrt[3]{yz}$ in exponential form.

$\sqrt[3]{yz} = \sqrt[3]{y}\sqrt[3]{z} = y^{\frac{1}{3}}z^{\frac{1}{3}}$.

Exponent Laws For Rational Exponents

Example

Evaluate $32^{\frac{1}{5}}$.

$32^{\frac{1}{5}} = \sqrt[5]{32} = 2$.

Example

Evaluate $\left(\frac{1}{81}\right)^{\frac{1}{4}}$.

$\left(\frac{1}{81}\right)^{\frac{1}{4}} = \frac{\sqrt[4]{1}}{\sqrt[4]{81}} = \frac{1}{3}$.

Exponent Laws For Rational Exponents

What happens if the numerator $\neq 1$?

Using earlier exponent laws, $x^{\frac{m}{n}} = \left(x^{\frac{1}{n}}\right)^m = \left(\sqrt[n]{x}\right)^m$.

By the same token, $x^{\frac{m}{n}} = (x^m)^{\frac{1}{n}} = \sqrt[n]{x^m}$.

Exponent Law When Numerator $\neq 1$

For any value x, $x^{\frac{m}{n}} = \left(\sqrt[n]{x}\right)^m = \sqrt[n]{x^m}$, provided that $x \geq 0$ if n is even.

The first form is generally preferrable since it makes values smaller before making them larger again.

Exponent Laws For Rational Exponents

Example

Write $x^{\frac{4}{3}}$ in radical form.

$x^{\frac{4}{3}} = \left(\sqrt[3]{x}\right)^4$, or $x^{\frac{4}{3}} = \sqrt[3]{x^4}$.

Example

Write $\sqrt[10]{x^3}$ in exponential form.

$\sqrt[10]{x^3} = x^{\frac{3}{10}}$.

Exponent Laws For Rational Exponents

Example

Evaluate $8^{\frac{2}{3}}$.

$8^{\frac{2}{3}} = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4$.

Example

Evaluate $(-25)^{\frac{3}{2}}$.

Since $(-25)^{\frac{3}{2}} = (\sqrt{-25})^3$, the answer is not a real number.

Questions?