MCR3U: Functions

Properties of Exponential Functions

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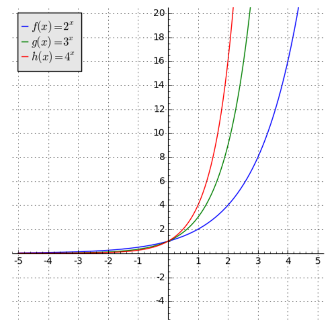
Properties of Exponential Functions

Consider the graphs of $f(x) = 2^x$, $g(x) = 3^x$ and $h(x) = 4^x$ below.



Properties of Exponential Functions

All exponential functions of the form $f(x) = b^x$, where $b > 0$, have the following features:

- the $f(x)$-intercept is at 1
- there is a horizontal asymptote at $y = 0$

The base, b, uniquely defines the overall shape of the graph.

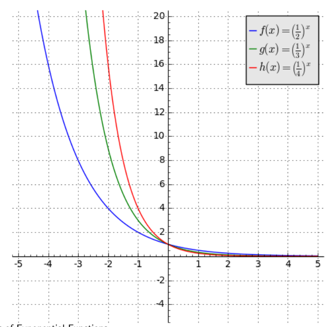
In particular, larger values of b cause $f(x)$ to increase more quickly.

Note the restrictions on b:

- if $b = 1$, $f(x) = 1$, which is not exponential
- if $b = 0$, $f(x) = 0$ for $x > 0$, and undefined for $x \leq 0$
- if $b < 0$, $f(x)$ is not defined, since it contains infinitely many vertical asymptotes between any two x values, when $x = \frac{m}{n}$ for even n

Properties of Exponential Functions

Now consider the graphs of $f(x) = \left(\frac{1}{2}\right)^x$, $g(x) = \left(\frac{1}{3}\right)^x$ and $h(x) = \left(\frac{1}{4}\right)^x$ below.



Properties of Exponential Functions

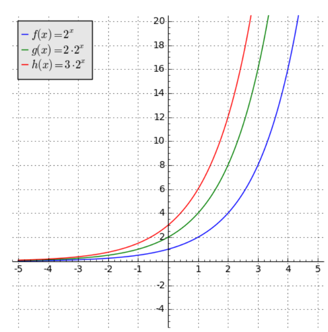
If $f(x)$ increases as x increases, then the function demonstrates *exponential growth*. This occurs when $b > 1$.

If $f(x)$ decreases as x increases, then the function demonstrates *exponential decay*. This occurs when $0 < b < 1$.

By recognizing exponential growth or decay, we can get an idea of the general shape of an exponential function.

Properties of Exponential Functions

Now consider the graphs of $f(x) = 2^x$, $g(x) = 2 \cdot 2^x$ and $h(x) = 3 \cdot 2^x$ below.



Properties of Exponential Functions

We can summarize the information about exponential functions of the form $f(x) = a \cdot b^x$.

Properties of Exponential Functions

If $f(x)$ is an exponential function of the form $f(x) = a \cdot b^x$, where $a \neq 0$ and $b > 0$, then:

- the $f(x)$-intercept is at a
- there is a horizontal asymptote at $y = 0$
- if $b > 1$, $f(x)$ grows exponentially
- if $0 < b < 1$, $f(x)$ decays exponentially

We will look at cases when $a < 0$ later, when we discuss transformations of exponential functions.

Properties of Exponential Functions

Example

State whether $f(x) = 7 \cdot 8^x$ represents exponential growth or decay.

Since $b > 1$, $f(x)$ represents exponential growth.

Example

State whether $f(x) = 6\left(\frac{5}{3}\right)^x$ represents exponential growth or decay.

As before, $b > 1$, so $f(x)$ represents exponential growth.

Properties of Exponential Functions

Example

State whether $f(x) = 4 \cdot 0.75^x$ represents exponential growth or decay.

Since $0 < b < 1$, $f(x)$ represents exponential decay.

Example

State whether $f(x) = \frac{3}{7} \cdot 2^x$ represents exponential growth or decay.

Since $b > 1$, $f(x)$ represents exponential growth. It does not matter that $0 < a < 1$.

Properties of Exponential Functions

Example

Sketch a graph of $f(x) = 2 \cdot 3^x$.

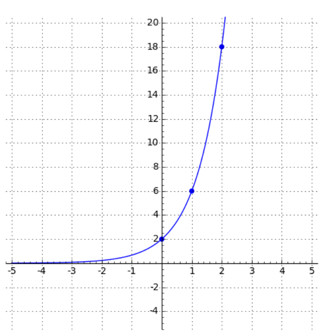
The $f(x)$-intercept is 2.

Since $b > 1$, the function demonstrates exponential growth.

There is a horizontal asymptote at $y = 0$.

Two other points on the graph are $(1, 6)$ and $(2, 18)$, since $f(1) = 6$ and $f(2) = 18$.

Properties of Exponential Functions



Properties of Exponential Functions

Example

Sketch a graph of $f(x) = 12\left(\frac{1}{2}\right)^x$.

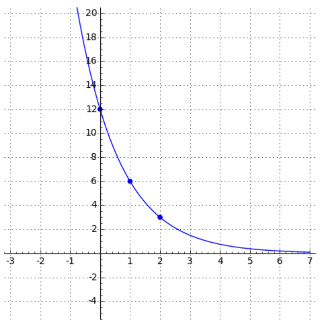
The $f(x)$-intercept is 12.

Since $0 < b < 1$, the function demonstrates exponential decay.

There is a horizontal asymptote at $y = 0$.

Two other points on the graph are $(1, 6)$ and $(2, 3)$, since $f(1) = 6$ and $f(2) = 3$.

Properties of Exponential Functions



Properties of Exponential Functions

Since an exponential function of the form $f(x) = a \cdot b^x$ involves repeated multiplication of the base b, all consecutive values of $f(x)$ will change by a factor of b.

Finite Differences for Exponential Functions

If $f(x)$ is an exponential function, then the ratio of any two consecutive finite differences is constant. This ratio is the base of the function.

This can be useful when classifying relations, or determining equations of exponential functions from a collection of known points.

Properties of Exponential Functions

Example

Which of the following relations is exponential?

x	y	$\Delta 1$
1	21	
2	34	13
3	47	13
4	60	13

x	y	$\Delta 1$
1	1	
2	7	6
3	17	10
4	31	14

x	y	$\Delta 1$
1	11	
2	19	8
3	35	16
4	67	32

Calculate the first differences for each relation.

The first relation is linear, the second is quadratic (the second differences are the constant 4), and the third is exponential (the ratio is $\frac{16}{8} = \frac{32}{16} = 2$).

Equations of Exponential Functions

Example

Determine an equation for an exponential function, in the form $f(x) = a \cdot b^x$, that passes through the points $(4, 3)$, $(5, 6)$ and $(6, 12)$.

Using the coordinates, calculate the first differences.

x	y	$\Delta 1$
4	3	
5	6	3
6	12	6

The ratio of first differences is $\frac{6}{3} = 2$, so $b = 2$.

Equations of Exponential Functions

Substitute a known point, such as $(4, 3)$, to determine the value of a.

$$\begin{aligned} 3 &= a \cdot 2^4 \\ 3 &= a \cdot 16 \\ a &= \tfrac{3}{16} \end{aligned}$$

Thus, an equation for the function is $f(x) = \frac{3}{16} \cdot 2^x$.

An alternate method of determining an equation involves solving a system of equations using exponent laws.

Equations of Exponential Functions

Example

An exponential function has the form $f(x) = a \cdot b^x$. If $f(3) = 1\,215$ and $f(7) = 15$, what is the value of $f(10)$?

Using the points $(3, 1\,215)$ and $(7, 15)$, set up the following two equations.

$$\begin{aligned} 1\,215 &= a \cdot b^3 \\ 15 &= a \cdot b^7 \end{aligned}$$

Isolate a in each equation.

$$\begin{aligned} a &= \frac{1\,215}{b^3} \\ a &= \frac{15}{b^7} \end{aligned}$$

Equations of Exponential Functions

Set the two right-hand sides equal to each other, and rearrange.

$$\frac{1\,215}{b^3} = \frac{15}{b^7}$$
$$\frac{b^7}{b^3} = \frac{15}{1\,215}$$

Simplify, using exponent laws.

$$b^4 = \frac{1}{81}$$
$$b^4 = \left(\tfrac{1}{3}\right)^4$$
$$b = \tfrac{1}{3}$$

Use $b = \frac{1}{3}$ to solve for a.

$$15 = a\left(\tfrac{1}{3}\right)^7$$
$$a = 32\,805$$

Equations of Exponential Functions

An equation for the function is $f(x) = 32\,805\left(\frac{1}{3}\right)^x$. Substitute $x = 10$.

$$f(10) = 32\,805\left(\frac{1}{3}\right)^{10}$$
$$= \tfrac{32\,805}{59\,049}$$
$$= \tfrac{5}{9}$$

Therefore, $f(10) = \frac{5}{9}$.

Questions?