MCR3U: Functions

Pascal's Triangle and the Binomial Theorem

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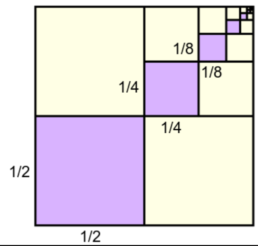

Pascal's Triangle

Pascal's Triangle is an arrangement of numbers, generated using a simple iterative process.

While Pascal's Triangle was not "invented" by Blaise Pascal, he is credited for applying it toward probability theory.

Begin with a triangular arrangement of 1s, as shown.

1

1 1

Continue to place 1s at the outer edges of each new row. Each interior value is the sum of the two values diagonally above it.

Pascal's Triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

etc.

Pascal's Triangle

The first row in Pascal's Triangle is Row 0. This means that the nth row has $n+1$ entries.

Each column is read diagonally downwards, from right to left. The first column is Column 0.

A entry's position may be denoted $t_{n,r}$, indicating the rth term in the nth row.

For example, $t_{4,1} = 4$ and $t_{5,3} = 10$, as shown on the previous slide.

Pascal's Triangle

$t_{0,0}$

$t_{1,0}$ $t_{1,1}$

$t_{2,0}$ $t_{2,1}$ $t_{2,2}$

$t_{3,0}$ $t_{3,1}$ $t_{3,2}$ $t_{3,3}$

$t_{4,0}$ $t_{4,1}$ $t_{4,2}$ $t_{4,3}$ $t_{4,4}$

$t_{5,0}$ $t_{5,1}$ $t_{5,2}$ $t_{5,3}$ $t_{5,4}$ $t_{5,5}$

etc.

Terms in Pascal's Triangle

For any term, $t_{n,r}$, in Pascal's Triangle,

$$t_{n,r} = \begin{cases} 1 & \text{if } r = 0 \text{ or } r = n \\ t_{n-1,r-1} + t_{n-1,r} & \text{otherwise} \end{cases}$$

Pascal's Triangle

Example

Express $t_{4,1} + t_{4,2}$ as a single term in Pascal's Triangle.

The term must be in the next row (Row 5) and the larger of the two columns (Column 2), so the term is $t_{5,2}$.

Example

Express $t_{16,7}$ as the sum of two terms in Pascal's Triangle.

The two terms that add to $t_{16,7}$ must come from the row above (Row 15), and the larger of the two columns will have the same value as the given term (7), so the sum is $t_{15,6} + t_{15,7}$.

Pascal's Triangle

There are many interesting patterns in Pascal's Triangle. One of these involves the sum of the entries in any given row.

In the first row, there is only a 1, so the sum is 1.

In the second row, $1 + 1 = 2$.

In the third row, $1 + 2 + 1 = 4$.

In the fourth row, $1 + 3 + 3 + 1 = 8$.

Sum of the Entries in a Row of Pascal's Triangle

The sum of all of the entries in the nth row of Pascal's Triangle is 2^n.

Pascal's Triangle

Another interesting pattern in Pascal's Triangle is often called "hockey stick" pattern.

Beginning at the first entry in any column, sum the numbers downward and left to some arbitrary point, then move down and right one entry.

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

Pascal's Triangle

The sum of the values in column 1 from $t_{1,1}$ to $t_{4,1}$ is $1 + 2 + 3 + 4 = 10$.

This value is the value of $t_{5,2}$.

"Hockey Stick" Pattern in Pascal's Triangle

In Pascal's Triangle, the sum of the first k entries in a column is $t_{n,r} + t_{n+1,r} + t_{n+2,r} + \ldots + t_{n+k,r} = t_{n+k+1,r+1}$.

There are many more patterns in Pascal's Triangle, but the main focus in this course is its application toward expanding binomials.

Binomial Theorem

Expand and simplify $(x + y)^2$.

$$\begin{aligned}
(x + y)^2 &= (x + y)(x + y) \\
&= x^2 + xy + xy + y^2 \\
&= 1x^2 + 2xy + 1y^2
\end{aligned}$$

Hmmmm. . .

Binomial Theorem

Expand and simplify $(x + y)^3$.

$$\begin{aligned}
(x + y)^3 &= (x + y)(x + y)^2 \\
&= (x + y)(x^2 + 2xy + y^2) \\
&= x^3 + 2x^2y + xy^2 + x^2y + 2xy^2 + y^3 \\
&= 1x^3 + 3x^2y + 3xy^2 + 1y^3
\end{aligned}$$

Looks familiar. . .

Binomial Theorem

Expand and simplify $(x + y)^4$.

$$(x + y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$$

Can you spot the pattern?

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}$$

Binomial Theorem

The coefficients of the simplified polynomial expression correspond to entries in Pascal's Triangle.

The coefficients in the expansion of $(x+y)^n$ are the entries in Row n.

The exponents follow a predictable pattern too.

The exponent for x decreases by 1, while the exponent for y increases by 1.

In each term, the sum of the exponents is n.

Binomial Theorem

$(x+y)^n = t_{n,0}x^ny^0 + t_{n,1}x^{n-1}y^1 + t_{n,2}x^{n-2}y^2 + \ldots + t_{n,n}x^0y^n$

Binomial Theorem

Example

Expand $(x+5)^3$.

Using Pascal's Triangle is faster than using the Distributive Property.

$$\begin{aligned}(x+5)^3 &= t_{3,0}x^3 \cdot 5^0 + t_{3,1}x^2 \cdot 5^1 + t_{3,2}x^1 \cdot 5^2 + t_{3,3}x^0 \cdot 5^3 \\ &= 1 \cdot x^3 + 3 \cdot 5x^2 + 3 \cdot 25x + 1 \cdot 125 \\ &= x^3 + 15x^2 + 75x + 125\end{aligned}$$

Binomial Theorem

Example

Expand $(3a+4b)^3$.

Use the substitutions $x=3a$, $y=4b$ and $n=3$.

$$\begin{aligned}(3a+4b)^3 &= t_{3,0}(3a)^3 + t_{3,1}(3a)^2(4b) + t_{3,2}(3a)(4b)^2 + t_{3,3}(4b)^3 \\ &= 1(27a^3) + 3(9a^2)(4b) + 3(3a)(16b^2) + 1(64b^3) \\ &= 27a^3 + 108a^2b + 144ab^2 + 64b^3\end{aligned}$$

Binomial Theorem

Example

Expand $(2a+3b)^4$.

Use the substitutions $x=2a$, $y=3b$ and $n=4$.

$$\begin{aligned}(2a+3b)^4 =& t_{4,0}(2a)^4 + t_{4,1}(2a)^3(3b) + t_{4,2}(2a)^2(3b)^2 \\ &+ t_{4,3}(2a)(3b)^3 + t_{4,4}(3b)^4 \\ =& 1(16a^4) + 4(8a^3)(3b) + 6(4a^2)(9b^2) \\ &+ 4(2a)(27b^3) + 1(81b^4) \\ =& 16a^4 + 96a^3b + 216a^2b^2 + 216ab^3 + 81b^4\end{aligned}$$

Binomial Theorem

Example

Expand $(4a-2b)^4$.

Use the substitutions $x=4a$, $y=-2b$ and $n=4$. Note that y is negative. What will happen?

$$\begin{aligned}(4a-2b)^4 =& t_{4,0}(4a)^4 + t_{4,1}(4a)^3(-2b) + t_{4,2}(4a)^2(-2b)^2 \\ &+ t_{4,3}(4a)(-2b)^3 + t_{4,4}(-2b)^4 \\ =& 1(256a^4) + 4(64a^3)(-2b) + 6(16a^2)(4b^2) \\ &+ 4(4a)(-8b^3) + 1(16b^4) \\ =& 256a^4 - 512a^3b + 384a^2b^2 - 128ab^3 + 16b^4\end{aligned}$$

If y is negative, the terms in the expansion alternate signs.

Questions?