MCR3U: Functions

Functions, Domain and Range

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Domain and Range

Consider the process of sketching a graph of $y = \sqrt{x - 5}$.

To do so, we would first need to determine appropriate values for x and calculate the resulting values of y.

The x-values are *input* into the equation. The set of all possible values for the independent variable is called the *domain*.

The y-values are *output* from the equation. The set of all possible values for the dependent variable is called the *range*.

A *relation* is a set of ordered pairs, associating values from the domain with those in the range.

Domain and Range

For the relation $y = \sqrt{x - 5}$, there are some restrictions on both the domain and the range.

Since the square root of a negative value is not defined in the real number system, the smallest possible value of x is 5, which will result in a y-value of 0.

Therefore, the domain of $y = \sqrt{x - 5}$ is any value of x greater than or equal to 5.

Since the relation will never be negative, the range of $y = \sqrt{x - 5}$ is any value of y greater than or equal to 0.

To express the domain and range mathematically, we can use one of two systems: *set-builder notation* or *interval notation*.

Set-Builder Notation

A *set* is a collection of objects (called *elements*), and both the domain and range are just sets of numbers.

To denote a set, we use curly brackets $\{\ldots\}$ with any elements, conditions or restrictions specified inside.

The first portion of set-builder notation typically specifies the number system to which the values belong. This is done via the $\in$ symbol, which is read "is in" or "belongs to".

For instance, to state that any number n belongs to the natural number system N, use $n \in N$.

Set-Builder Notation

The second portion of set-builder notation states the conditions and/or restrictions on the variable.

To indicate that all values of n are greater than 3, we could simply state $n > 3$.

To specify all values between 2 and 6 inclusive, we could state $2 \leq n \leq 6$.

To indicate that only odd numbers are permitted, a more complicated expression like $n = 2k + 1, k \geq 0$ may be needed.

Putting things together, the domain and range of $y = \sqrt{x - 5}$ could be stated as follows:

$$\text{Domain: } \{x \in R \mid x \geq 5\}$$
$$\text{Range: } \{y \in R \mid y \geq 0\}$$

Set-Builder Notation

Example

State the domain and range for the relation $y = x^2 - 4$.

This quadratic relation opens upward, and has its vertex located at $(0, -4)$.

The domain is the set of all real numbers, since any value of x may be input.

The range is the set of all real numbers greater than or equal to the minimum value, which is -4.

Therefore, the domain and range of $y = x^2 - 4$ are:

$$\text{Domain: } \{x \in R\}$$
$$\text{Range: } \{y \in R \mid y \geq -4\}$$

Interval Notation

An alternative to set-builder notation, interval notation specifies the intervals on which the domain or range of a relation is defined.

Interval notation uses two types of brackets. Square brackets indicate that a value is included in an interval, while round brackets exclude the value.

Since many relations continue toward infinity, the symbols ∞ and $-\infty$ are often used. As neither is an actual value, round brackets are always used with infinity.

If necessary, the "union" symbol, $\cup$, can be used to join together two intervals that are split by one or more values.

Interval Notation

Using interval notation, the domain and range of $y = x^2 - 4$ are:

$$\text{Domain: } (-\infty, \infty)$$
$$\text{Range: } [-4, \infty)$$

Since there is no restriction on the domain, any value of x between $-\infty$ and ∞ can be used.

Don't be confused by the two different types of brackets in the range. In this case, -4 is included in the interval, while ∞ is not.

Domain and Range

Example

State the domain and range of $y = \frac{1}{x-2}$ using both set-builder and interval notation.

Since we cannot divide by zero, x cannot be 2. Thus, the domain is the set of all real numbers *except* for 2.

It is not possible for y to be 0, although it can be very close. Thus, the range is the set of all real numbers *except* for 0.

Using the two systems:

$$\text{Domain: } \{x \in R \mid x \neq 2\} \quad \text{Domain: } (-\infty, 2) \cup (2, \infty)$$
$$\text{Range: } \{y \in R \mid y \neq 0\} \quad \text{Range: } (-\infty, 0) \cup (0, \infty)$$

Domain and Range

Sometimes, a relation can be specified by listing all ordered pairs, such as when using a table of values.

In this case, the domain and range can be specified as sets of values.

Consider the relation $R = \{(1,5), (0,-3), (7,1), (-4,-9)\}$.

The domain is simply $\{1, 0, 7, -4\}$ and the range is $\{5, -3, 1, -9\}$.

Note that we cannot say, for example, that the domain is all values between -4 and 7, since the relation only uses the four *discrete* points listed.

Functions and Relations

In addition to being relations, all of the previous examples were *functions*.

A function is a relation where each element in its domain corresponds to exactly one element in its range.

For example, in the relation $R = \{(1,5), (0,-3), (7,1), (-4,-9)\}$, the value of 1 in the domain is associated with a value of 5 in the range.

If the relation also contained the point $(1, 10)$, then the relation would not be a function.

When examining data, look for duplication in the range for each specific value in the domain.

Functions and Relations

Example

Determine whether the relation $R = \{(1,3), (4,3), (7,-2), (10,0)\}$ is a function.

The relation is a function, since each element in the domain is associated with exactly one element in the range.

Example

Determine whether the relation $R = \{(5,1), (2,-3), (5,8), (-1,7)\}$ is a function.

The relation is not a function, since the value of 5 in the domain is associated with two values, 1 and 8, in the range.

Functions and Relations

Another way to determine whether a relation is a function is to graph it.

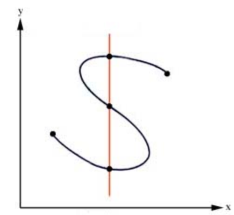
Since each element of the domain must be associated with exactly one element in the range, we can attempt to draw a vertical line on the graph such that it intersects the relation at two or more points.

This is known as the *Vertical Line Test*.

Functions and Relations

Vertical Line Test

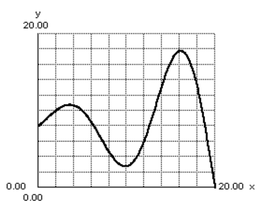
If it is possible to draw a vertical line anywhere along a graph, such that the vertical line intersects the graph more than once, then at least one element in the domain is associated with more than one value in the range, and the graph is not a function.



Functions and Relations

Example

Determine whether the graph below is a function.

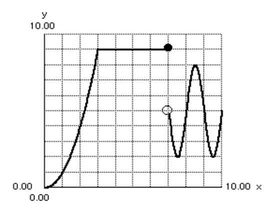


Since any vertical line drawn would not intersect the graph more than once, the relation is a function.

Functions and Relations

Example

Determine whether the graph below is a function.



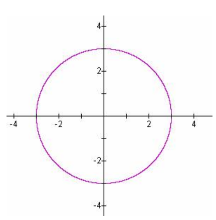
Again, a vertical line would not intersect more than once. Remember that an open dot means “not included” while a closed dot means “included”.

Functions and Relations

Your Turn

Determine whether the relation $x^2 + y^2 = 9$ is a function.

The relation is a circle, centred at the origin, with radius 3.



Since it is possible to intersect the graph more than once using a vertical line, the relation is not a function.

Functions and Relations

The last example could also be solved algebraically by isolating y and seeing if more than one answer is possible.

$$\begin{aligned} x^2 + y^2 &= 9 \\ y^2 &= 9 - x^2 \\ y &= \pm\sqrt{9 - x^2} \end{aligned}$$

Remember that when inverting a square, both the positive and negative square roots are necessary.

Since there are values of x in the domain that result in more than one value of y, the relation is not a function.

Functions and Relations

Example

Determine whether the relation $x = (y-2)^2 - 5$ is a function.

$$\begin{aligned} x &= (y-2)^2 - 5 \\ x + 5 &= (y-2)^2 \\ \pm\sqrt{x+5} &= y - 2 \\ 2 \pm \sqrt{x+5} &= y \end{aligned}$$

Again, the relation is not a function. The graph of this particular function looks like a parabola turned on its side. We will see this again shortly.

Questions?