MCR3U: Functions

Exact Values of Trigonometric Ratios

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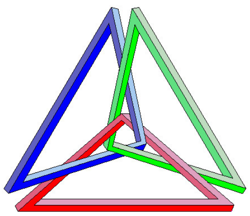
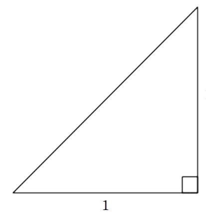


Special Angles

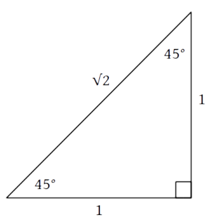
Consider a right-isosceles triangle where the two equal sides have a length of 1 unit.



Since an isosceles triangle has two equal angles, the acute angles must both be 45°.

Special Angles

Using the Pythagorean Theorem for the hypotenuse, $h = \sqrt{1^2 + 1^2} = \sqrt{2}$.



The resulting triangle can be used to determine exact values for trigonometric ratios involving 45°.

Special Angles

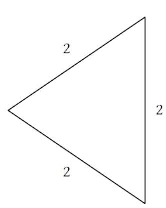
$$\sin 45^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \qquad \cos 45^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \qquad \tan 45^\circ = \frac{1}{1} = 1$$

and

$$\csc 45^\circ = \frac{\sqrt{2}}{1} = \sqrt{2} \qquad \sec 45^\circ = \frac{\sqrt{2}}{1} = \sqrt{2} \qquad \cot 45^\circ = \frac{1}{1} = 1$$

Special Angles

Now consider an equilateral triangle with side lengths of 2 units.

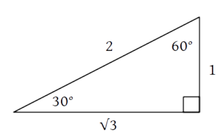


An equilateral triangle contains three 60° angles.

Special Angles

An altitude from one vertex creates two congruent right-triangles with 30° and 60° angles, one side 1 unit long, and a hypotenuse 2 units long.

Using the Pythagorean Theorem for the other side, $a^2 = \sqrt{2^2 - 1^2} = \sqrt{3}$.



This triangle can be used to find exact values for trigonometric ratios involving 30° or 60°.

Special Angles

$\sin 30° = \frac{1}{2}$ $\quad$ $\cos 30° = \frac{\sqrt{3}}{2}$ $\quad$ $\tan 30° = \frac{\sqrt{3}}{3}$

and

$\csc 30° = 2$ $\quad$ $\sec 30° = \frac{2\sqrt{3}}{3}$ $\quad$ $\cot 30° = \sqrt{3}$

$\sin 60° = \frac{\sqrt{3}}{2}$ $\quad$ $\cos 60° = \frac{1}{2}$ $\quad$ $\tan 60° = \sqrt{3}$

and

$\csc 60° = \frac{2\sqrt{3}}{3}$ $\quad$ $\sec 60° = 2$ $\quad$ $\cot 60° = \frac{\sqrt{3}}{3}$

Angles in Quadrant 1

Recall that two angles are coterminal if their terminal arms are in the same position on the coordinate plane.

For example, 405° is coterminal with 45°, since $405° - 360° = 45°$.

This means that the ratios for 405° are the same as those for 45°.

For instance, $\sin 405° = \sin 45° = \frac{\sqrt{2}}{2}$.

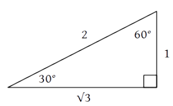
Negative rotations may also result in coterminal angles in Q1.

Angles in Quadrant 1

Example

Determine the exact value of $\sin -330°$.

A rotation of $-330°$ is the same as a rotation of 30°.



Using the 30-60-90 triangle, the opposite side has a length of 1 and the hypotenuse has a length of 2.

Therefore, $\sin 30° = \frac{\text{opp}}{\text{hyp}} = \frac{1}{2}$.

Angles in Quadrants 2-4

When an angle lies in quadrants 2-4, we use the reference angle – the acute angle made with the x-axis.

By the CAST rule, trigonometric ratios may be positive or negative, depending on the quadrant in which the angle falls.

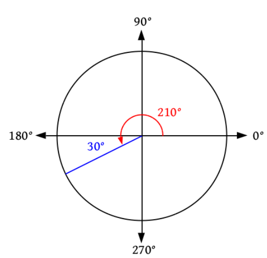
When used in conjunction with the CAST rule, it is possible to determine the exact values of trigonometric ratios in the other three quadrants, including their sign.

Angles in Quadrants 2-4

Example

Determine the exact value of $\sin 210°$.

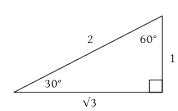
210° falls in quadrant 3, with a reference angle of 30°.



Angles in Quadrants 2-4

Since sine is negative in Q3, the exact value will be too.

Therefore, $\sin 210° = -\sin 30°$.



Using the 30-60-90 triangle, $\sin 30° = \frac{1}{2}$.

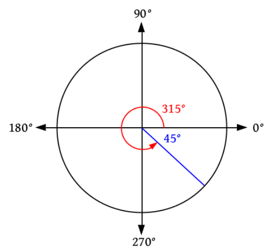
So, $\sin 210° = -\frac{1}{2}$.

Angles in Quadrants 2-4

Example

Determine the exact value of $\cos 315°$.

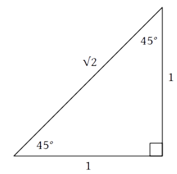
$315°$ falls in quadrant 4, with a reference angle of $45°$.



Angles in Quadrants 2-4

Since cosine is positive in Q4, the exact value will be too.

Therefore, $\cos 315° = \cos 45°$.



Using the 45-45-90 triangle, $\cos 45° = \frac{\sqrt{2}}{2}$.

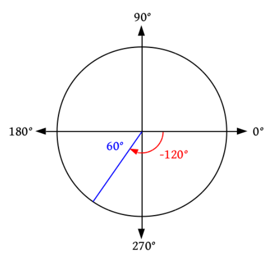
So, $\cos 315° = \frac{\sqrt{2}}{2}$.

Angles in Quadrants 2-4

Example

Determine the exact value of $\csc(-120°)$.

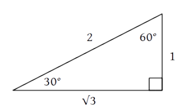
$-120°$ falls in quadrant 3, with a reference angle of $60°$.



Angles in Quadrants 2-4

Since cosecant is negative in Q3, the exact value will be too.

Therefore, $\csc(-120°) = -\csc 60°$.



Using the 30-60-90 triangle, $\csc 60° = \frac{2\sqrt{3}}{3}$.

So, $\csc(-120°) = -\frac{2\sqrt{3}}{3}$.

Angles Falling Between Quadrants

Sometimes, the terminal arm of an angle falls on an axis, between two quadrants.

The following table summarizes the exact values of the primary trigonometric ratios for $0°$, $90°$, $180°$ and $270°$.

Angle	$0°$	$90°$	$180°$	$270°$
$\sin\theta$	0	1	0	-1
$\cos\theta$	1	0	-1	0
$\tan\theta$	0	undef.	0	undef.

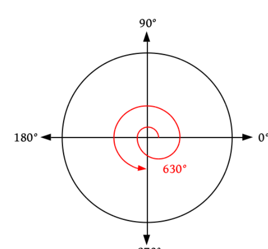
These values will take on more meaning in the next unit on trigonometric functions.

Angles Falling Between Quadrants

Example

Determine the exact value of $\cos 630°$.

Since $630°$ is coterminal with $270°$ as shown, $\cos 630° = \cos 270° = 0$.



Questions?