MCR3U: Functions

Coterminal Angles / Ratios for Any Angle

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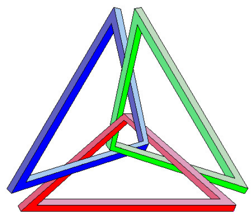
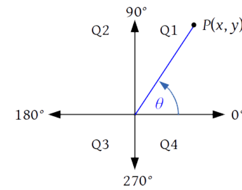


Angles On the Coordinate Plane

Imagine the following angle drawn on the coordinate plane, passing through point P.

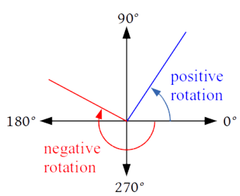


The line through point P and the origin that forms an angle, θ, with the x-axis is called the *terminal arm*.

Angles On the Coordinate Plane

An angle may be expressed using either a *positive rotation*, or a *negative rotation*.

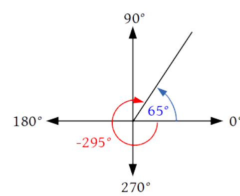
Positive rotations begin at 0° and rotate counter-clockwise about the origin, while negative rotations rotate clockwise.



Coterminal Angles

Two angles are *coterminal* if their terminal arms are in the same position on the coordinate plane.

Both positive and negative rotations may result in coterminal angles.

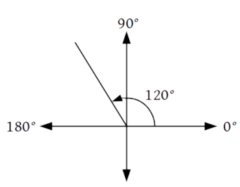


Coterminal Angles

Example

State two angles (one positive, one negative) that are coterminal with 120°.

The terminal arm of an angle of 120° falls in Q2 as shown.



Coterminal Angles

To find a positive angle coterminal with 120°, add 360°.

$$120^\circ + 360^\circ = 480^\circ$$

To find a negative angle coterminal with 120°, subtract 360°.

$$120^\circ - 360^\circ = -240^\circ$$

Additional angles can be found by adding or subtracting again.

Other coterminal angles are 840°, 1200°, -600°, and so on.

Coterminal Angles

Example

Determine the angle, $0^\circ \leq \theta \leq 360^\circ$, that is coterminal with 815°.

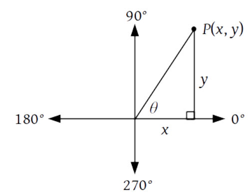
Since $815^\circ \div 360^\circ \approx 2.3$, an angle of 815° has undergone approximately 2.3 rotations.

Subtract two full rotations of 360° to obtain the angle required.

$$\begin{aligned} \theta &= 815^\circ - 2 \times 360^\circ \\ &= 95^\circ \end{aligned}$$

Trigonometric Ratios For Any Angle

If we specify a point through which the terminal arm passes, we can construct a right triangle relative to the x-axis.



The x-coordinate is the horizontal component of the triangle, while the y-coordinate is the vertical component.

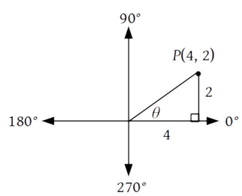
From this, we can work out the values of the six trigonometric ratios.

Trigonometric Ratios For Any Angle

Example

Determine the values of the six trigonometric ratios for an angle whose terminal arm passes through $(4, 2)$.

Draw a diagram to visualize.



Trigonometric Ratios For Any Angle

Use the Pythagorean Theorem to calculate the length of the hypotenuse.

$$\begin{aligned} h^2 &= 4^2 + 2^2 \\ &= 20 \\ h &= \sqrt{20} \\ &= 2\sqrt{5} \end{aligned}$$

Now that all three sides are known, we can state the six ratios.

Trigonometric Ratios For Any Angle

$$\sin\theta = \frac{2}{2\sqrt{5}} = \frac{\sqrt{5}}{5} \qquad \cos\theta = \frac{4}{2\sqrt{5}} = \frac{2\sqrt{5}}{5} \qquad \tan\theta = \frac{2}{4} = \frac{1}{2}$$

and

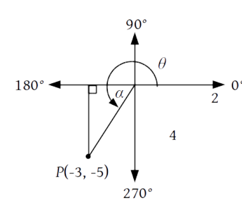
$$\csc\theta = \frac{2\sqrt{5}}{2} = \sqrt{5} \qquad \sec\theta = \frac{2\sqrt{5}}{4} = \frac{\sqrt{5}}{2} \qquad \cot\theta = \frac{4}{2} = 2$$

Trigonometric Ratios For Any Angle

Example

Determine the values of the six trigonometric ratios for an angle whose terminal arm passes through $(-3, -5)$.

Draw a diagram to visualize.



Trigonometric Ratios For Any Angle

Rather than using θ, we can use the *reference angle* α instead.

This is because if the triangle was "mirrored" in Q1, the acute angle θ would have the same value as α.

As before, use the Pythagorean Theorem to calculate the length of the hypotenuse.

$$\begin{aligned} h^2 &= (-3)^2 + (-5)^2 \\ &= 34 \\ h &= \sqrt{34} \end{aligned}$$

Since the terminal arm is in quadrant 3, the negative x- and y-coordinates will affect the ratios.

Trigonometric Ratios For Any Angle

$$\sin\theta = \frac{-5}{\sqrt{34}} = -\frac{5\sqrt{34}}{34} \qquad \cos\theta = \frac{-3}{\sqrt{34}} = -\frac{3\sqrt{34}}{34} \qquad \tan\theta = \frac{-5}{-3} = \frac{5}{3}$$

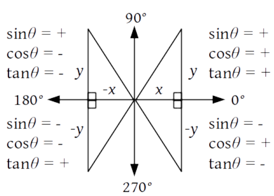
and

$$\csc\theta = \frac{\sqrt{34}}{-5} = -\frac{\sqrt{34}}{5} \qquad \sec\theta = \frac{\sqrt{34}}{-3} = -\frac{\sqrt{34}}{3} \qquad \cot\theta = \frac{-3}{-5} = \frac{3}{5}$$

CAST Rule

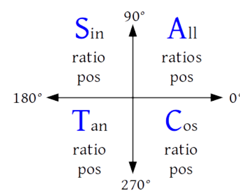
In the last example, the tangent ratio (as well as its reciprocal, cotangent) was positive in Q3, while the sine and cosine ratios (and their reciprocals) were negative.

The following diagram states the sign of the primary trigonometric ratios in all four quadrants.



CAST Rule

A mnemonic used to remember which ratio is positive in a given quadrant is the CAST Rule.



By using the CAST rule, we can quickly determine whether a trigonometric ratio for a given angle is positive or negative, based on the quadrant in which the angle falls.

CAST Rule

In every 360° rotation, there are two quadrants in which a specific angle is positive, and two in which it is negative.

This means that if we are given a ratio for some angle θ, where $0^\circ \leq \theta \leq 360^\circ$, then θ may have two possible values.

For example, $\sin 30^\circ = \frac{1}{2}$ (Q1), but $\sin 150^\circ = \frac{1}{2}$ (Q3) as well.

This is because the reference angle α in Q3 is $180^\circ - 150^\circ = 30^\circ$.

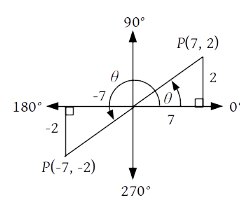
If we know that $\sin\theta = \frac{1}{2}$, where $0^\circ \leq \theta \leq 360^\circ$, then θ may be either 30° or 150°.

CAST Rule

Example

Determine the value of θ, $0^\circ \leq \theta \leq 360^\circ$, if $\tan\theta = \frac{2}{7}$.

Since $\tan\theta$ is positive, θ falls in either Q1 or Q3.



CAST Rule

The angle in Q1 has a measure of $\tan^{-1}\left(\frac{2}{7}\right) \approx 16°$.

Since θ is acute, it is also the reference angle in Q3.

Therefore, the angle in Q3 has a measure of $180° + 16° \approx 196°$.

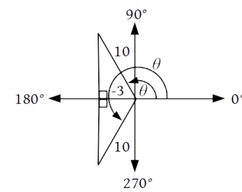
Thus, θ is either $16°$ or $196°$.

CAST Rule

Example

Determine the value of θ, $0° \leq \theta \leq 180°$, if $\cos\theta = -\frac{3}{10}$.

Since $\cos\theta$ is negative, θ falls in either Q2 or Q3.



CAST Rule

Since $0° \leq \theta \leq 180°$, we can reject the angle in Q3.

The angle in Q2 has a measure of $\cos^{-1}\left(-\frac{3}{10}\right) \approx 107°$.

If θ *could* have been in Q3, its measure could have been caucluated using one of two methods.

- adding the reference angle, $180° - 107° \approx 73°$, to $180°$: $180° + 73° \approx 253°$
- using a negative rotation from $360°$: $360° - 107° \approx 253°$

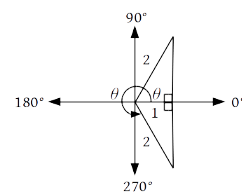
As it stands, the only solution is $\theta \approx 107°$.

CAST Rule

Example

If $\cos\theta = \frac{1}{2}$ and $0° \leq \theta \leq 360°$, determine $\sin\theta$.

Since $\cos\theta$ is positive, θ falls in either Q1 or Q4.



Trigonometric Ratios For Any Angle

Consider the case in quadrant 1 first. Using the Pythagorean Theorem to calculate the length of the opposite side, y, we obtain $y = \sqrt{2^2 - 1^2} = \sqrt{3}$.

In quadrant 1, $\sin\theta$ is positive, so $\tan\theta = \frac{\sqrt{3}}{1} = \sqrt{3}$.

In quadrant 4, $\sin\theta$ is negative, so $\tan\theta = \frac{-\sqrt{3}}{1} = -\sqrt{3}$.

Both ratios are solutions to the question, since both satisfy the conditions.

Questions?