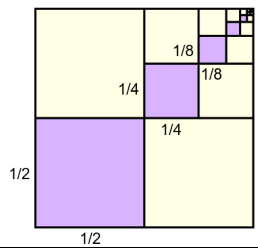
MCR3U: Functions

Arithmetic Series

J. Garvin



Arithmetic Series

Recall that a sequence is an ordered set of values.

A *series* is the sum the terms in a sequence.

Like sequences, some series are finite while others are infinite.

In this course, we will focus on the sum of a finite number of terms.

Consider the arithmetic sequence $3, 7, 11, 15, \ldots$. If we were to add each term, $3 + 7 + 11 + 15 + \ldots$, we would have an *arithmetic series*.

Arithmetic Series

If a sequence is arithmetic, then the sum of its terms is an arithmetic series.

Arithmetic Series

What is the sum of the positive integers 1 through 100?

While it is possible to calculate this sum by adding together all 100 integers, there is a much faster method.

The sum, S, of the integers is $1 + 2 + 3 + \ldots + 50 + 51 + \ldots + 98 + 99 + 100$.

Rearranging the terms of this arithmetic series, we obtain the following:

$$\begin{aligned} S &= \underbrace{(1+100)+(2+99)+(3+98)+\ldots+(50+51)}_{50 \text{ terms}} \\ &= 50 \cdot 101 \\ &= 5\,050 \end{aligned}$$

Arithmetic Series

Sum of the First n Positive Integers

The sum of the first n positive integers, S_n, is given by $S_n = \frac{n}{2} \cdot (1 + n)$.

Anecdotal evidence suggests that Carl Friedrich Gauss developed this formula in response to a teacher's punishment while he was a boy, but this may not be entirely accurate.

Example

What is the sum of the first 18 923 integers?

Clearly, this would take too long to add each term.

$$\begin{aligned} S_{18\,923} &= \frac{18\,923}{2} \cdot (1 + 18\,923) \\ &= 179\,049\,426 \end{aligned}$$

Arithmetic Series

The previous formula can be generalized to any arithmetic series $t_1 + t_2 + t_3 + \ldots + t_n$.

Create the following pairs, working inward from the outer terms:

$$S = \underbrace{(t_1 + t_n) + (t_2 + t_{n-1}) + (t_3 + t_{n-2}) + \ldots}_{\frac{n}{2} \text{ terms}}$$

Since there is a common difference d, all terms are "evenly spaced". Therefore, all pairs in the above series are equal.

Sum of n Terms in an Arithmetic Series

For an arithmetic series, the sum of the first n terms, S_n, is given by $S_n = \frac{n}{2} \cdot (t_1 + t_n)$ where t_1 and t_n are the values of the first and last terms.

Arithmetic Series

Example

Determine the sum of the first 20 terms of the series $13 + 19 + 25 + \ldots$.

First, calculate t_{20} using the formula for the general term of an arithmetic sequence.

$$\begin{aligned} t_{20} &= 13 + (20 - 1)(6) \\ &= 127 \end{aligned}$$

Now use the formula for an arithmetic series to calculate S_{20}.

$$\begin{aligned} S_{20} &= \frac{20}{2} \cdot (13 + 127) \\ &= 1\,400 \end{aligned}$$

The sum of the first 20 terms is 1 400.

Arithmetic Series

Example

Determine the sum of the first 35 terms of the series $8 - 3 - 14 - \ldots$.

This is still an arithmetic series. The terms are negative because $d = -11$.

$$\begin{aligned} t_{35} &= 8 + (35 - 1)(-11) \\ &= -366 \end{aligned}$$

Substitute into the formula for an arithmetic series.

$$\begin{aligned} S_{35} &= \frac{35}{2} \cdot (8 - 366) \\ &= -6\,265 \end{aligned}$$

The sum of the first 35 terms is $-6\,265$.

Arithmetic Series

Example

Determine the sum of the series $34 + 56 + \ldots + 826$.

First we need to determine the number of terms, n.

$$\begin{aligned} 826 &= 34 + (n - 1)(22) \\ 792 &= (n - 1)(22) \\ 36 &= n - 1 \\ n &= 37 \end{aligned}$$

Now we can use the arithmetic series formula as before.

$$\begin{aligned} S_{37} &= \frac{37}{2} \cdot (34 + 826) \\ &= 15\,910 \end{aligned}$$

Thus, the sum of the series is $15\,910$.

Arithmetic Series

Example

The sum of the first 24 terms in an arithmetic series is $3\,588$. Determine the value of the 10th term, if the 1st term is 46.

Use $n = 24$, $S_{24} = 3\,588$, and $t_1 = 46$ to solve for t_{24}.

$$\begin{aligned} 3\,588 &= \frac{24}{2} \cdot (46 + t_n) \\ 299 &= 46 + t_n \\ t_n &= 253 \end{aligned}$$

Solve for d using the general term of an arithmetic sequence.

$$\begin{aligned} 253 &= 46 + (24 - 1)d \\ d &= 9 \end{aligned}$$

Arithmetic Series

Finally, solve for t_{10}.

$$\begin{aligned} t_{10} &= 46 + (10 - 1)(9) \\ &= 127 \end{aligned}$$

Therefore, the 10th term is 127.

Arithmetic Series

Example

A seating area in a theatre contains 38 rows of chairs. Each row contains 2 more chairs than the row before it. If there are 16 chairs in the first row, how many chairs are in the entire seating area?

Use the formula for the general term to calculate t_{38}.

$$\begin{aligned} t_{38} &= 16 + (38 - 1)(2) \\ &= 90 \end{aligned}$$

Use the formula for an arithmetic series to find S_{38}.

$$\begin{aligned} S_{38} &= \frac{38}{2} \cdot (16 + 90) \\ &= 2\,014 \end{aligned}$$

There are $2\,014$ seats in the seating area.

Questions?