MCR3U: Functions

Applications of Exponential Functions

Part 2: More Complicated Problems

J. Garvin

Applications of Exponential Functions

We have seen how certain phenomena can be modelled using exponential functions of the form $f(t) = a \cdot b^{t/p}$.

In most cases, we are given four of the five values directly, and need to solve for the remaining value.

In other cases, it is straightforward to work out a required value which can then be used.

There are other times, however, when the values are not so obvious, or when an alternate form for the model is required.

Applications of Exponential Functions

Example

The intensity of light decreases by 2% every metre below the surface of a lake. A diver with an underwater camera requires at least 50% of the light at the surface to use it. Will he be able to photograph some fish swimming 20 metres below the surface of the water?

Since the amount of light decreases as depth increases, this is an example of exponential decay.

If $L(d)$ represents the percentage of light at depth d, then the scenario can be modelled using the equation $L(d) = a \cdot b^d$.

It may seem like there is not enough information given, namely the values of a, b and $L(d)$ are not specified.

Applications of Exponential Functions

Since there is 100% of the light at the surface, then $a = 100$.

Since the light decreases by 2% per metre, $b = 1 - 0.02 = 0.98$. This can be thought of as "98% of the light remains for every extra metre of depth".

Therefore, the equation becomes $L(d) = 100 \cdot 0.98^d$.

At 20 m, $L(20) = 100 \cdot 0.98^{20} \approx 66.76\%$.

Therefore, the diver will be able to use his camera.

Applications of Exponential Functions

Example

What is the maximum depth at which the diver can use the camera?

Use the previous equation, with $L(d) = 50$.

$$50 = 100 \cdot 0.98^d$$
$$\tfrac{1}{2} = 0.98^d$$

Using "guess and check", $d \approx 34.3$ m. At any greater depth, the camera will not work properly.

Applications of Exponential Functions

Example

A container of 56°C water is placed in a freezer, kept a constant temperature of −8°C. If the temperature of the soup is halved every 10 minutes, how long will it take for the water to freeze?

This situation is different from the previous example, because the minimum temperature of the water is −8°C, the temperature of the freezer.

This implies a horizontal asymptote at $y = -8$, resulting in an equation of the form $T(t) = a \cdot b^{t/p} - 8$.

The initial value, then, will be $56 + 8 = 64$.

Since the temperature is halved every 10 minutes, $b = \frac{1}{2}$ and $p = 10$.

Applications of Exponential Functions

An equation, then, is $T(t) = 64\left(\frac{1}{2}\right)^{t/10} - 8$.

Since water freezes at 0°C, we want to find the value of t when $T(t) = 0$.

$$\begin{aligned} 0 &= 64\left(\tfrac{1}{2}\right)^{t/10} - 8 \\ 8 &= 64\left(\tfrac{1}{2}\right)^{t/10} \\ \tfrac{1}{8} &= \left(\tfrac{1}{2}\right)^{t/10} \\ \left(\tfrac{1}{2}\right)^{3} &= \left(\tfrac{1}{2}\right)^{t/10} \\ 3 &= \tfrac{t}{10} \\ t &= 30 \end{aligned}$$

The water begins to freeze after half an hour.

Applications of Exponential Functions

Example

The spread of an invasive plant can be modelled using an exponential function. In 2012, the plant covered 24 km^2 of land, while in 2015 it covered 375 km^2. Predict how much land will be covered in 2020.

Two data points are $(2012, 24)$ and $(2015, 375)$. Both a and b can be determined using a system of equations.

$$\begin{aligned} 24 &= a \cdot b^{2012} \\ 375 &= a \cdot b^{2015} \end{aligned}$$

Rearrange for a.

$$\begin{aligned} a &= \frac{24}{b^{2012}} \\ a &= \frac{375}{b^{2015}} \end{aligned}$$

Applications of Exponential Functions

Equate the right-hand sides and solve for b.

$$\begin{aligned} \frac{24}{b^{2012}} &= \frac{375}{b^{2015}} \\ \frac{b^{2015}}{b^{2012}} &= \tfrac{375}{24} \\ b^3 &= \tfrac{375}{24} \\ b^3 &= \tfrac{125}{8} \\ b &= \tfrac{5}{2} \end{aligned}$$

If $b = \frac{5}{2}$, then $a = \frac{24}{\left(\frac{5}{2}\right)^{2012}}$.

Unfortunately, $a \approx 5.3 \times 10^{-800}$, which is far too impractical to work with. We need an alternative method.

Applications of Exponential Functions

If we let 2012 represent the starting year, then we have the following information:

- the initial year is 0, and the final year is $2015 - 2012 = 3$, so $t = 3$
- the initial population is $a = 24$ and the final population is $P(t) = 375$
- the base, b, is unknown

Substiting these values, we obtain the same value of b as earlier.

$$\begin{aligned} 375 &= 24 \cdot b^3 \\ \tfrac{125}{8} &= b^3 \\ b &= \tfrac{5}{2} \end{aligned}$$

Applications of Exponential Functions

Therefore, the plant population can be modelled by $P(t) = 24\left(\frac{5}{2}\right)^{t}$.

Since we want to know the population in 2020, the time elapsed since 2012 is 8 years.

Plug $t = 8$ into the equation.

$$\begin{aligned} P(8) &= 24\left(\frac{5}{2}\right)^{8} \\ &= 36\,621 \end{aligned}$$

Thus, the plant would cover an area of approximately 36 621 km^2, or roughly the entire island of Taiwan.

Questions?