MCR3U: Functions

Simple Annuities

Part 2: Present Value

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Annuities

Recap

How much needs to be invested each week into an account paying 3.1%/a interest, compounded weekly, to be worth \$20 000 in 15 years? How much was invested? How much interest was earned?

Determine the regular payment, R, that will grow to \$20 000.

$$R = \frac{20\,000 \cdot \frac{0.031}{52}}{\left(1+\frac{0.031}{52}\right)^{15\times52} - 1}$$
$$\approx \$20.15$$

A total of $15 \times 52 \times \$20.15 \approx \$15\,717$ was invested, earning $\$20\,000 - \$15\,717 \approx \$4\,283$ in interest.

Annuities

When dealing with the future value of an annuity, we are interested in the amount to which the investment will grow given regular deposits for some fixed time.

Some times, we are interested in knowing how much money must be invested into an account to provide regular *withdrawals* for some amount of time.

This is known as the *present value* of an annuity – what do we need *now* to provide for future income.

When all withdrawals have been made from the account, the balance will be zero, or very close to it.

Annuities

Consider the case where some amount of money is deposited into an account that pays i%/a interest over n years. This account will provide for regular withdrawals of \$$R$ each year.

Since the first withdrawal is made at the end of the first year, the amount of money needed to grow to \$$R$ is $\frac{R}{(1+i)}$.

The second withdrawal occurs at the end of the second year, so there are two years to earn interest. The amount that must be deposited for this withdrawal is $\frac{R}{(1+i)^2}$.

The amount needed for the third withdrawal is $\frac{R}{(1+i)^3}$, etc.

Annuities

The present value of the annuity is the sum of the values in the series $\frac{R}{(1+i)} + \frac{R}{(1+i)^2} + \frac{R}{(1+i)^3} + \ldots + \frac{R}{(1+i)^n}$.

This is a geometric series, with an initial term of R and a common ratio of $\frac{1}{(1+i)}$.

Using the formula for a geometric series, we can determine the present value of the annuity.

$$P = R \cdot \frac{\left(\frac{1}{(1+i)}\right)^n - 1}{\frac{1}{(1+i)} - 1}$$
$$\ldots$$
$$= R \cdot \frac{1-(1+i)^{-n}}{i}$$

Annuities

Present Value of a Simple Annuity, Comp. Annually

The present value of an annuity, with regular withdrawals of \$$R$ from an account paying i%/a interest, for n years, is given by $P = R \cdot \frac{1-(1+i)^{-n}}{i}$.

As before, we can specify the compounding frequency, c, if it is not annual.

Present Value of a Simple Annuity, Comp. Regularly

The present value of an annuity, with regular withdrawals of \$$R$ from an account paying i%/a interest, with compounding frequency c, for n years, is given by $P = R \cdot \frac{1-\left(1+\frac{i}{c}\right)^{-nc}}{\frac{i}{c}}$.

Annuities

Example

How much should be invested into an account paying 5%/a interest, compounded monthly, to provide for 10 years of regular monthly withdrawals of \$1 500?

$$P = 1\,500 \cdot \frac{1-\left(1+\frac{0.05}{12}\right)^{-10\times12}}{\frac{0.05}{12}}$$
$$\approx \$141\,422.03$$

Example

How much was withdrawn from the account, and how much was interest?

There were $10 \times 12 = 120$ withdrawals, for a total of $120 \times \$1\,500 = \$180\,000$. \$38 577.97 was interest.

Annuities

Example

If \$25 000 is invested into an account paying 4.5%/a interest, compounded bi-monthly, how much money can be withdrawn from the account every two months for the next 6 years?

Rearrange the equation to solve for R.

$$25\,000 = R \cdot \frac{1-\left(1+\frac{0.045}{6}\right)^{-6\times6}}{\frac{0.045}{6}}$$
$$R = \frac{25\,000 \cdot \frac{0.045}{6}}{1-\left(1+\frac{0.045}{6}\right)^{-36}}$$
$$\approx \$794.99$$

Just less than \$800 can be withdrawn every two months.

Annuities

Example

If \$100 000 is donated to a university to pay for \$7 000 in scholarships each year. If the money is invested in an account paying 3.7%/a interest, compounded annually, for how many years will the donation pay out?

Rearrange the equation to solve for n.

$$100\,000 = 7\,000 \cdot \frac{1-(1+0.037)^{-n}}{0.037}$$
$$1 - \frac{100\,000 \cdot 0.037}{7\,000} = \frac{1}{(1.037)^n}$$
$$0.471 \approx 0.964^n$$

Through trial-and-error, $n \approx 20.5$, so the donation will pay for 20 years of scholarships.

Annuities

Example

At age 38, Ben decides to plan out his retirement. He estimates that he will need \$2 400 each month for 25 years, beginning at age 65. He plans to fund his retirement by making regular bi-weekly contributions into an account paying 3.5%/a interest, compounded bi-weekly. His withdrawals will come from a "safer" account that pays 2.1%/a interest, compounded monthly. How much should he contribute every two weeks in order to achieve this goal?

To answer this question, we must first find the present value of the annuity that will provide the monthly withdrawals.

Once the present value is known, we can use this as the future value of an annuity, and solve for the regular payment.

Annuities

Start with the present value of the annuity.

$$P = 2\,400 \cdot \frac{1-\left(1+\frac{0.021}{12}\right)^{-25\times12}}{\frac{0.021}{12}}$$
$$\approx \$559\,780.31$$

Use this amount as the future value of annuity, paid into for 31 years.

$$R = \frac{559\,780.31 \cdot \frac{0.035}{26}}{\left(1+\frac{0.035}{26}\right)^{27\times26} - 1}$$
$$\approx \$479.61$$

To achieve his retirement goal, Ben should deposit just over \$493 every two weeks.

Questions?