MCR3U: Functions

Simple Annuities

Part 1: Future Value

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Compound Interest

Recap

How much needs to be invested into an account paying 5.4%/a interest, compounded semi-monthly, to be worth \$6 500 in 4 years?

Use $FV = \$6\,500$, $i = 0.054$, $n = 4$, and $c = 24$.

$$\begin{aligned} P &= \frac{6\,500}{\left(1+\frac{0.054}{24}\right)^{4\times 24}} \\ &\approx \$5\,238.55 \end{aligned}$$

Annuities

All of the previous examples involving compound interest involved investing some amount of money and letting it accumulate interest over some span of time.

While many investments are made this way, it is not always practical (or possible) to invest a large sum of money and simply let it grow.

Most people invest money *as they earn it*. That is, they may invest \$100 into a savings account one week, then \$300 the next, then \$50 the next, and so on.

An investment in which contributions are made on more than one occasion is called an *annuity*.

An investment in which regular contributions are made according to a fixed schedule is called a *simple annuity*.

Annuities

Consider the case of investing some regular amount, R, into an account paying i%/a interest over n years.

After the first contribution, there will be a total of \$$R$ in the account.

After the second contribution, there will be an additional \$$R$ in the account. But the first \$$R$ will have earned interest according to the compound interest equation.

Therefore, the account will be worth $\$R + \$R(1+i)$.

After the third contribution, there will be an additional \$$R$ deposited while the previous two contributions will earn interest.

The account will be worth $\$R + \$R(1+i) + \$R(1+i)^2$.

Annuities

After all n contributions have been made, the account will be worth $\$R + \$R(1+i) + \$R(1+i)^2 + \ldots + \$R(1+i)^{n-1}$.

This is a geometric series, with an initial term of R and a common ratio of $(1+i)$.

Therefore, we can use the formula for a geometric series to determine the future value of the annuity.

$$\begin{aligned} FV &= R \cdot \frac{(1+i)^n - 1}{(1+i) - 1} \\ &= R \cdot \frac{(1+i)^n - 1}{i} \end{aligned}$$

Annuities

Future Value of a Simple Annuity, Compounded Annually

The future value of an annuity, with regular payments of \$$R$ into an account paying i%/a interest, for n years, is given by $FV = R \cdot \frac{(1+i)^n - 1}{i}$.

Like the formula for compound interest, we can specify the compounding frequency, c, if it is not annual.

Future Value of a Simple Annuity, Compounded Regularly

The future value of an annuity, with regular payments of \$$R$ into an account paying i%/a interest, with compounding frequency c, for n years, is given by $FV = R \cdot \frac{(1+\frac{i}{c})^{nc} - 1}{\frac{i}{c}}$.

Annuities

Example

At the end of each month, \$100 is deposited into an account paying 4%/a interest, compounded monthly. What is the investment worth after 5 years?

$$FV = 100 \cdot \frac{\left(1+\frac{0.04}{12}\right)^{5\times12}-1}{\frac{0.04}{12}}$$
$$\approx \$6\,629.90$$

Example

How much was deposited into the account, and how much interest was earned?

There were $5 \times 12 = 60$ deposits, for a total of $60 \times \$100 = \$6\,000$. The interest earned was \$629.90.

Annuities

Example

Bert and Ernie both want to retire at age 65. Bert starts saving at age 25, depositing \$200 from each bi-weekly paycheque into an account paying \$2.8%/a interest, compounded bi-weekly. Ernie starts later, at age 45, depositing \$500 every two weeks into a similar account. Describe each investment, when they reach age 65.

Begin with Bert's investment.

$$FV = 200 \cdot \frac{\left(1+\frac{0.028}{26}\right)^{40\times26}-1}{\frac{0.028}{26}}$$
$$\approx \$383\,130.01$$

Annuities

Compare this with Ernie's investment.

$$FV = 500 \cdot \frac{\left(1+\frac{0.028}{26}\right)^{20\times26}-1}{\frac{0.028}{26}}$$
$$\approx \$348\,281.64$$

By the time they are 65, Bert's investment is worth nearly \$35 000 more than Ernie's.

Over his 40 years of payments, Bert deposited $40 \times 26 \times \$200 = \$208\,000$, earning \$175 130.01 in interest.

Over his 20 years of payments, Ernie deposited $20 \times 26 \times \$500 = \$260\,000$, earning \$88 281.64 in interest.

So, Bert not only has more money when he retires, but he also deposits less money than Ernie by investing sooner.

Annuities

Example

To save for \$8 000 university tuition, a student starts investing money each month into an account paying 3.2%/a interest, compounded monthly. How much will the student need to deposit each month to reach this goal in 4 years?

We need to determine the regular payment, R.

$$8\,000 = R \cdot \frac{\left(1+\frac{0.032}{12}\right)^{4\times12}-1}{\frac{0.032}{12}}$$
$$R = \frac{8\,000 \cdot \frac{0.032}{12}}{\left(1+\frac{0.032}{12}\right)^{4\times12}-1}$$
$$\approx \$156.45$$

The student will meet the goal by depositing $\sim$ \$160/mo.

Questions?